Appenndix

数値計算に必要な内容を書こうと思います。

連立方程式の解

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

$$x=\frac{d_1-b_1y-c_1z}{a_1}$$

$$a_2\frac{d_1-b_1y-c_1z}{a_1}+b_2y+c_2z=d_2$$

$$a_3\frac{d_1-b_1y-c_1z}{a_1}+b_3y+c_3z=d_3$$

$$a_2d_1-a_2b_1y-c_1a_2z+a_1b_2y+c_2a_1z=a_1d_2$$

$$a_3d_1-a_3b_1y-c_1a_3z+a_1b_3y+c_3a_1z=a_1d_3$$

$$(a_1b_2-a_2b_1)y+(c_2a_1-c_1a_2)z=a_1d_2-a_2d_1$$

$$(a_1b_3-a_3b_1)y+(c_3a_1-c_1a_3)z=a_1d_3-a_3d_1$$

$$y=\frac{a_1d_2-a_2d_1}{a_1b_2-a_2b_1}-\frac{c_2a_1-c_1a_2}{a_1b_2-a_2b_1}z$$

$$y=\frac{a_1d_3-a_3d_1}{a_1b_3-a_3b_1}-\frac{c_3a_1-c_1a_3}{a_1b_3-a_3b_1}z$$

$$(\frac{c_3a_1-c_1a_3}{a_1b_3-a_3b_1}-\frac{c_2a_1-c_1a_2}{a_1b_2-a_2b_1})z=\frac{a_1d_3-a_3d_1}{a_1b_3-a_3b_1}-\frac{a_1d_2-a_2d_1}{a_1b_2-a_2b_1}$$

$$z=\frac{(a_1b_2-a_2b_1)(a_1d_3-a_3d_1)-(a_1b_3-a_3b_1)(a_1d_2-a_2d_1)}{(a_1b_2-a_2b_1)(c_3a_1-c_1a_3)-(a_1b_3-a_3b_1)(c_2a_1-c_1a_2)}$$

$$=\frac{a_1a_1b_2d_3+a_2a_3b_1d_1+a_1a_2b_3d_1+a_1a_3b_1d_2-a_1a_3b_2d_1-a_1a_2b_1d_3-a_1a_1b_3d_2-a_2a_3b_1d_1}{a_1a_1b_2c_3+a_2a_3b_1c_1+a_1a_2b_3c_1+a_1a_3b_1c_2-a_1a_3b_2c_1-a_1a_2b_1c_3-a_1a_1b_3c_2-a_2a_3b_1c_1}$$

$$=\frac{a_1a_1b_2d_3+a_1a_2b_3d_1+a_1a_3b_1d_2-a_1a_3b_2d_1-a_1a_2b_1d_3-a_1a_1b_3d_2}{a_1a_1b_2c_3+a_1a_2b_3c_1+a_1a_3b_1c_2-a_1a_3b_2c_1-a_1a_2b_1c_3-a_1a_1b_3c_2}$$

$$=\frac{a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_2b_1d_3-a_1b_3d_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

$$(a_1b_2-a_2b_1)y+(c_2a_1-c_1a_2)z=a_1d_2-a_2d_1$$
$$(a_1b_3-a_3b_1)y+(c_3a_1-c_1a_3)z=a_1d_3-a_3d_1$$

$$\frac{a_1b_2-a_2b_1}{c_2a_1-c_1a_2}y+z=\frac{a_1d_2-a_2d_1}{c_2a_1-c_1a_2}$$
$$\frac{a_1b_3-a_3b_1}{c_3a_1-c_1a_3}y+z=\frac{a_1d_3-a_3d_1}{c_3a_1-c_1a_3}$$

$$\frac{a_1b_2-a_2b_1}{c_2a_1-c_1a_2}y-\frac{a_1b_3-a_3b_1}{c_3a_1-c_1a_3}y=\frac{a_1d_2-a_2d_1}{c_2a_1-c_1a_2}-\frac{a_1d_3-a_3d_1}{c_3a_1-c_1a_3}$$

$$\{(c_3a_1-c_1a_3)(a_1b_2-a_2b_1)-(c_2a_1-c_1a_2)(a_1b_3-a_3b_1)\}y$$
$$=(c_3a_1-c_1a_3)(a_1d_2-a_2d_1)-(c_2a_1-c_1a_2)(a_1d_3-a_3d_1)$$

$$y=\frac{(c_3a_1-c_1a_3)(a_1d_2-a_2d_1)-(c_2a_1-c_1a_2)(a_1d_3-a_3d_1)}{(c_3a_1-c_1a_3)(a_1b_2-a_2b_1)-(c_2a_1-c_1a_2)(a_1b_3-a_3b_1)}$$
$$=\frac{c_3a_1a_1d_2-c_3a_1a_2d_1-c_1a_3a_1d_2+c_1a_3a_2d_1-c_2a_1a_1d_3+c_2a_1a_3d_1+c_1a_2a_1d_3-c_1a_2a_3d_1}{c_3a_1a_1b_2-c_3a_1a_2b_1-c_1a_3a_1b_2+c_1a_3a_2b_1-c_2a_1a_1b_3+c_2a_1a_3b_1+c_1a_2a_1b_3-c_1a_2a_3b_1}$$
$$=\frac{c_3a_1d_2-c_3a_2d_1-c_1a_3d_2-c_2a_1d_3+c_2a_3d_1+c_1a_2d_3}{c_3a_1b_2-c_3a_2b_1-c_1a_3b_2-c_2a_1b_3+c_2a_3b_1+c_1a_2b_3}$$
$$=\frac{a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_2d_1c_3-a_1d_3c_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

$$x=\frac{d_1-b_1y-c_1z}{a_1}$$
$$=\frac{\begin{aligned}&d_1(a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2)\\&-b_1(a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_2d_1c_3-a_1d_3c_2)\\&-c_1(a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_2b_1d_3-a_1b_3d_2)\end{aligned}}{a_1(a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2)}$$
$$=\frac{\begin{aligned}&a_1b_2c_3d_1+a_2b_3c_1d_1+a_3b_1c_2d_1-a_3b_2c_1d_1-a_2b_1c_3d_1-a_1b_3c_2d_1\\&-a_1b_1c_3d_2-a_2b_1c_1d_3-a_3b_1c_2d_1+a_3b_1c_1d_2+a_2b_1c_3d_1+a_1b_1c_2d_3\\&-a_1b_2c_1d_3-a_2b_3c_1d_1-a_3b_1c_1d_2+a_3b_2c_1d_1+a_2b_1c_1d_3+a_1b_3c_1d_2\end{aligned}}{a_1(a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2)}$$
$$=\frac{b_1c_2d_3+b_2c_3d_1+b_3c_1d_2-b_3c_2d_1-b_2c_1d_3-b_1c_3d_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

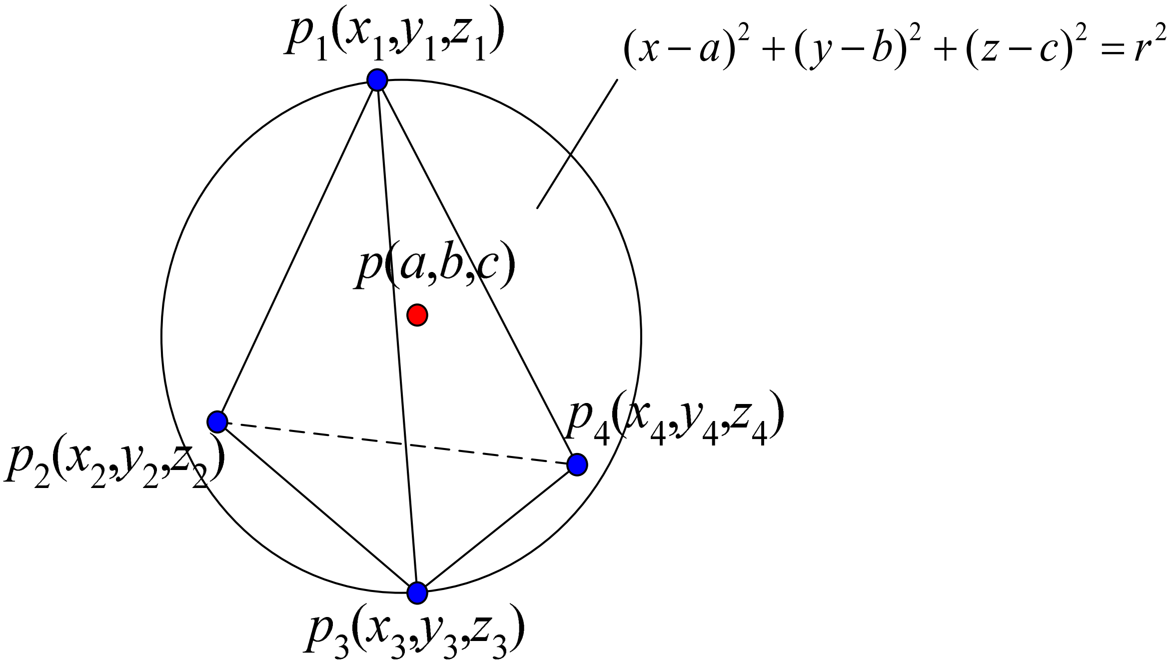
$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

$$x = \frac{b_1c_2d_3 + b_2c_3d_1 + b_3c_1d_2 - b_3c_2d_1 - b_2c_1d_3 - b_1c_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}$$

4 面体の外接球の中心座標と半径を計算します。



外接球の式に節点の座標を代入すると次式が得られます。

$$\begin{cases}(a-x_1)^2+(b-y_1)^2+(c-z_1)^2=r^2\\(a-x_2)^2+(b-y_2)^2+(c-z_2)^2=r^2\\(a-x_3)^2+(b-y_3)^2+(c-z_3)^2=r^2\\(a-x_4)^2+(b-y_4)^2+(c-z_4)^2=r^2\end{cases}$$

半径 r を消すと

$$\begin{cases}(a-x_2)^2+(b-y_2)^2+(c-z_2)^2=(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\\(a-x_3)^2+(b-y_3)^2+(c-z_3)^2=(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\\(a-x_4)^2+(b-y_4)^2+(c-z_4)^2=(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\end{cases}$$

展開して

$$\begin{cases}-2x_2a+x_2^2-2y_2b+y_2^2-2z_2c+z_2^2=-2x_1a+x_1^2-2y_1b+y_1^2-2z_1c+z_1^2\\-2x_3a+x_3^2-2y_3b+y_3^2-2z_3c+z_3^2=-2x_1a+x_1^2-2y_1b+y_1^2-2z_1c+z_1^2\\-2x_4a+x_4^2-2y_4b+y_4^2-2z_4c+z_4^2=-2x_1a+x_1^2-2y_1b+y_1^2-2z_1c+z_1^2\end{cases}$$

係数ごとにまとめて

$$\begin{cases}2(x_1-x_2)a+2(y_1-y_2)b+2(z_1-z_2)c=x_1^2-x_2^2+y_1^2-y_2^2+z_1^2-z_2^2\\2(x_1-x_3)a+2(y_1-y_3)b+2(z_1-z_3)c=x_1^2-x_3^2+y_1^2-y_3^2+z_1^2-z_3^2\\2(x_1-x_4)a+2(y_1-y_4)b+2(z_1-z_4)c=x_1^2-x_4^2+y_1^2-y_4^2+z_1^2-z_4^2\end{cases}$$

ここで、3元1次連立方程式の解は次式で与えられました。

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

$$x=\frac{b_1c_2d_3+b_2c_3d_1+b_3c_1d_2-b_3c_2d_1-b_2c_1d_3-b_1c_3d_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

$$y=\frac{a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_2d_1c_3-a_1d_3c_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

$$z=\frac{a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_2b_1d_3-a_1b_3d_2}{a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2}$$

従って、今回計算する式に当てはめると

$$a_1=2(x_1-x_2), b_1=2(y_1-y_2), c_1=2(z_1-z_2), d_1=x_1^2-x_2^2+y_1^2-y_2^2+z_1^2-z_2^2$$

$$a_2=2(x_1-x_3), b_2=2(y_1-y_3), c_2=2(z_1-z_3), d_2=x_1^2-x_3^2+y_1^2-y_3^2+z_1^2-z_3^2$$

$$a_3=2(x_1-x_4), b_3=2(y_1-y_4), c_3=2(z_1-z_4), d_3=x_1^2-x_4^2+y_1^2-y_4^2+z_1^2-z_4^2$$

これを解くと次式が導かれます。

分母

$$\begin{aligned}
&a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2 \\
&=8(x_1y_3z_4+x_1y_4z_2+x_1y_2z_3+x_2y_1z_4+x_2y_3z_1+x_2y_4z_3 \\
&+x_3y_1z_2+x_3y_4z_1+x_3y_2z_4+x_4y_1z_3+x_4y_2z_1+x_4y_3z_2 \\
&-x_1y_3z_2-x_1y_2z_4-x_1y_4z_3-x_2y_3z_4-x_2y_1z_3-x_2y_4z_1 \\
&-x_3y_4z_2-x_3y_1z_4-x_3y_2z_1-x_4y_2z_3-x_4y_1z_2-x_4y_3z_1)
\end{aligned}$$

x座標の分子

$$b_1c_2d_3 + b_2c_3d_1 + b_3c_1d_2 - b_3c_2d_1 - b_2c_1d_3 - b_1c_3d_2$$

$$= 4($$

$$+ y_2z_3x_1^2 + y_3z_4x_1^2 + y_4z_2x_1^2 + y_3z_1x_2^2 + y_4z_3x_2^2 + y_1z_4x_2^2$$

$$+ y_1z_2x_3^2 + y_2z_4x_3^2 + y_4z_1x_3^2 + y_2z_1x_4^2 + y_3z_2x_4^2 + y_1z_3x_4^2$$

$$+ y_2z_3y_1^2 + y_3z_4y_1^2 + y_4z_2y_1^2 + y_3z_1y_2^2 + y_4z_3y_2^2 + y_1z_4y_2^2$$

$$+ y_1z_2y_3^2 + y_2z_4y_3^2 + y_4z_1y_3^2 + y_2z_1y_4^2 + y_3z_2y_4^2 + y_1z_3y_4^2$$

$$+ y_2z_3z_1^2 + y_3z_4z_1^2 + y_4z_2z_1^2 + y_3z_1z_2^2 + y_4z_3z_2^2 + y_1z_4z_2^2$$

$$+ y_1z_2z_3^2 + y_2z_4z_3^2 + y_4z_1z_3^2 + y_2z_1z_4^2 + y_3z_2z_4^2 + y_1z_3z_4^2$$

$$- y_3z_2x_1^2 - y_4z_3x_1^2 - y_2z_4x_1^2 - y_1z_3x_2^2 - y_3z_4x_2^2 - y_4z_1x_2^2$$

$$- y_2z_1x_3^2 - y_4z_2x_3^2 - y_1z_4x_3^2 - y_1z_2x_4^2 - y_2z_3x_4^2 - y_3z_1x_4^2$$

$$- y_3z_2y_1^2 - y_4z_3y_1^2 - y_2z_4y_1^2 - y_1z_3y_2^2 - y_3z_4y_2^2 - y_4z_1y_2^2$$

$$- y_2z_1y_3^2 - y_4z_2y_3^2 - y_1z_4y_3^2 - y_1z_2y_4^2 - y_2z_3y_4^2 - y_3z_1y_4^2$$

$$- y_3z_2z_1^2 - y_4z_3z_1^2 - y_2z_4z_1^2 - y_1z_3z_2^2 - y_3z_4z_2^2 - y_4z_1z_2^2$$

$$- y_2z_1z_3^2 - y_4z_2z_3^2 - y_1z_4z_3^2 - y_1z_2z_4^2 - y_2z_3z_4^2 - y_3z_1z_4^2$$

$$)$$

y 座標の分子

$$a_1d_2c_3+a_2d_3c_1+a_3d_1c_2-a_3d_2c_1-a_2d_1c_3-a_1d_3c_2$$

$$=4($$

$$+x_4z_3x_1^2+x_3z_2x_1^2+x_2z_4x_1^2+x_3z_4x_2^2+x_4z_1x_2^2+x_1z_3x_2^2$$

$$+x_2z_1x_3^2+x_1z_4x_3^2+x_4z_2x_3^2+x_1z_2x_4^2+x_2z_3x_4^2+x_3z_1x_4^2$$

$$-x_2z_3x_1^2-x_3z_4x_1^2-x_4z_2x_1^2-x_4z_3x_2^2-x_3z_1x_2^2-x_1z_4x_2^2$$

$$-x_3z_2x_4^2-x_2z_1x_4^2-x_1z_3x_4^2-x_1z_2x_3^2-x_2z_4x_3^2-x_4z_1x_3^2$$

$$+x_1z_2y_4^2+x_2z_3y_4^2+x_3z_1y_4^2+x_3z_4y_2^2+x_4z_1y_2^2+x_1z_3y_2^2$$

$$+x_2z_1y_3^2+x_1z_4y_3^2+x_4z_2y_3^2+x_4z_3y_1^2+x_3z_2y_1^2+x_2z_4y_1^2$$

$$-x_2z_3y_1^2-x_3z_4y_1^2-x_4z_2y_1^2-x_4z_3y_2^2-x_3z_1y_2^2-x_1z_4y_2^2$$

$$-x_1z_2y_3^2-x_2z_4y_3^2-x_4z_1y_3^2-x_2z_1y_4^2-x_1z_3y_4^2-x_3z_2y_4^2$$

$$+x_3z_2z_1^2+x_2z_4z_1^2+x_4z_3z_1^2+x_3z_4z_2^2+x_4z_1z_2^2+x_1z_3z_2^2$$

$$+x_2z_1z_3^2+x_1z_4z_3^2+x_4z_2z_3^2+x_1z_2z_4^2+x_2z_3z_4^2+x_3z_1z_4^2$$

$$-x_1z_2z_3^2-x_2z_4z_3^2-x_4z_1z_3^2-x_3z_2z_4^2-x_2z_1z_4^2-x_1z_3z_4^2$$

$$-x_2z_3z_1^2-x_3z_4z_1^2-x_4z_2z_1^2-x_4z_3z_2^2-x_3z_1z_2^2-x_1z_4z_2^2$$

$$)$$

z 座標の分子

$$a_1b_2d_3+a_2b_3d_1+a_3b_1d_2-a_3b_2d_1-a_2b_1d_3-a_1b_3d_2$$

$$=4($$

$$+x_2y_3x_1^2+x_3y_4x_1^2+x_4y_2x_1^2+x_4y_3x_2^2+x_3y_1x_2^2+x_1y_4x_2^2$$

$$+x_1y_2x_3^2+x_2y_4x_3^2+x_4y_1x_3^2+x_3y_2x_4^2+x_2y_1x_4^2+x_1y_3x_4^2$$

$$-x_2y_4x_1^2-x_3y_2x_1^2-x_4y_3x_1^2-x_3y_4x_2^2-x_4y_1x_2^2-x_1y_3x_2^2$$

$$-x_2y_1x_3^2-x_4y_2x_3^2-x_1y_4x_3^2-x_1y_2x_4^2-x_2y_3x_4^2-x_3y_1x_4^2$$

$$+x_2y_3y_1^2+x_3y_4y_1^2+x_4y_2y_1^2+x_4y_3y_2^2+x_3y_1y_2^2+x_1y_4y_2^2$$

$$+x_1y_2y_3^2+x_2y_4y_3^2+x_4y_1y_3^2+x_3y_2y_4^2+x_2y_1y_4^2+x_1y_3y_4^2$$

$$-x_2y_4y_1^2-x_3y_2y_1^2-x_4y_3y_1^2-x_3y_4y_2^2-x_4y_1y_2^2-x_1y_3y_2^2$$

$$-x_2y_1y_3^2-x_4y_2y_3^2-x_1y_4y_3^2-x_1y_2y_4^2-x_2y_3y_4^2-x_3y_1y_4^2$$

$$+x_2y_3z_1^2+x_3y_4z_1^2+x_4y_2z_1^2+x_4y_3z_2^2+x_3y_1z_2^2+x_1y_4z_2^2$$

$$+x_1y_2z_3^2+x_2y_4z_3^2+x_4y_1z_3^2+x_3y_2z_4^2+x_2y_1z_4^2+x_1y_3z_4^2$$

$$-x_2y_4z_1^2-x_3y_2z_1^2-x_4y_3z_1^2-x_3y_4z_2^2-x_4y_1z_2^2-x_1y_3z_2^2$$

$$-x_2y_1z_3^2-x_4y_2z_3^2-x_1y_4z_3^2-x_1y_2z_4^2-x_2y_3z_4^2-x_3y_1z_4^2$$

$$)$$

・座標変換

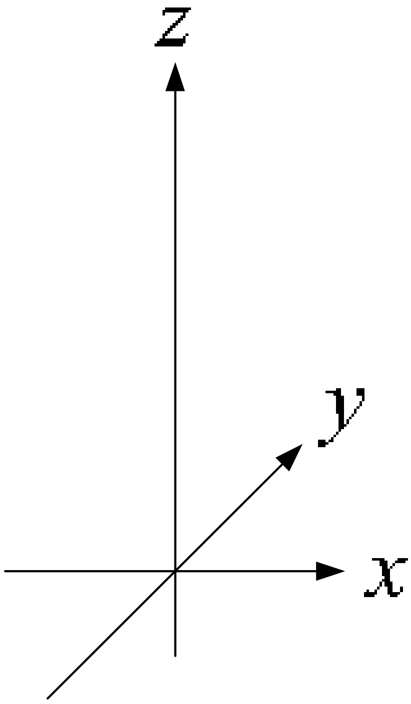
座標変換について書きます。数値計算では、直交座標以外にも円筒座標（円柱座標）、球座標（極座標）で解析が行われることがあります。理由は、例えば水滴や円筒型のロケットを解析する場合は、円筒座標や球座標を用いた方がメッシュの作成が容易になるからです。

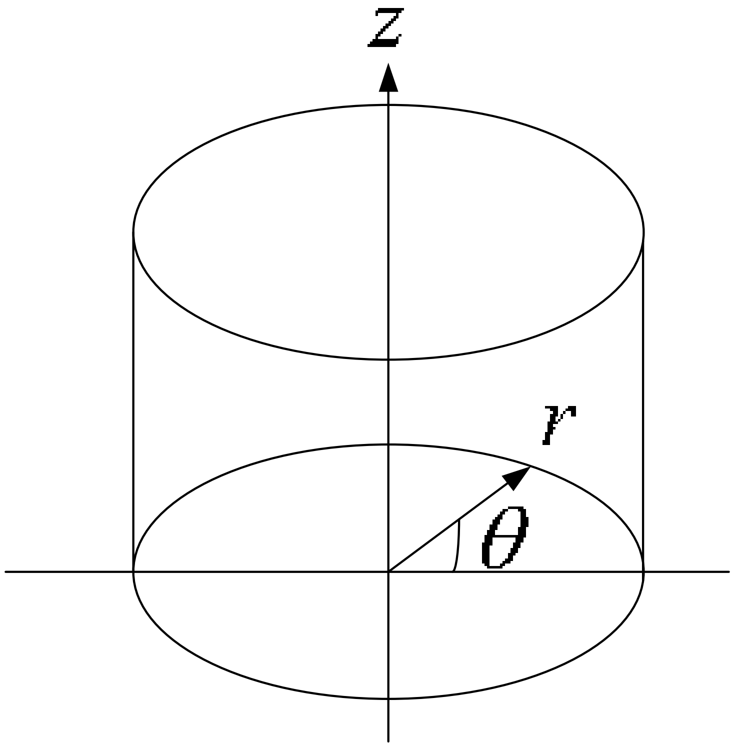
・円筒座標（円柱座標）

直交座標と円筒座標の座標変換について述べます。変数はそれぞれ下記が用いられます。

直交座標　(x, y, z)

円筒座標　(r, θ, z)

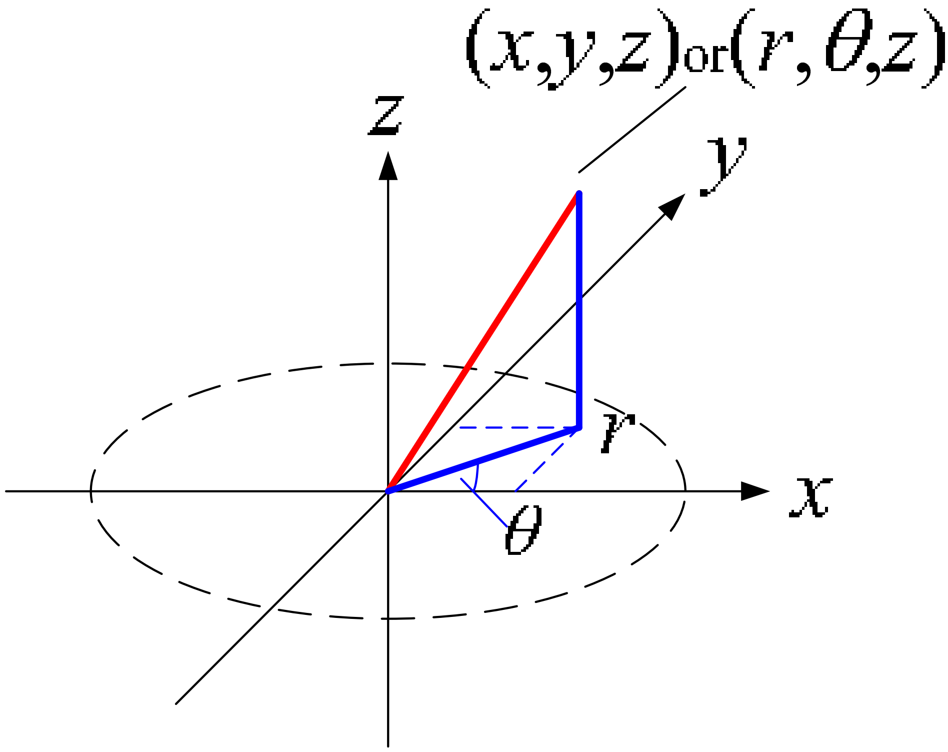




円筒座標(r, θ, z)⇒直交座標(x, y, z)

円筒座標を直交座標に変換する場合です。変数は下記で表されます。

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



ベクトルの変換は次式となります。

$$\vec{\delta}_x = \vec{\delta}_r(\cos\theta) + \vec{\delta}_\theta(-\sin\theta) + \vec{\delta}_z(0)$$

$$\vec{\delta}_y = \vec{\delta}_r(\sin\theta) + \vec{\delta}_\theta(\cos\theta) + \vec{\delta}_z(0)$$

$$\vec{\delta}_z = \vec{\delta}_r(0) + \vec{\delta}_\theta(0) + \vec{\delta}_z(1)$$

作用素の変換は次式となります。

$$\frac{\partial}{\partial x} = (\cos\theta)\frac{\partial}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial}{\partial\theta} + (0)\frac{\partial}{\partial z}$$

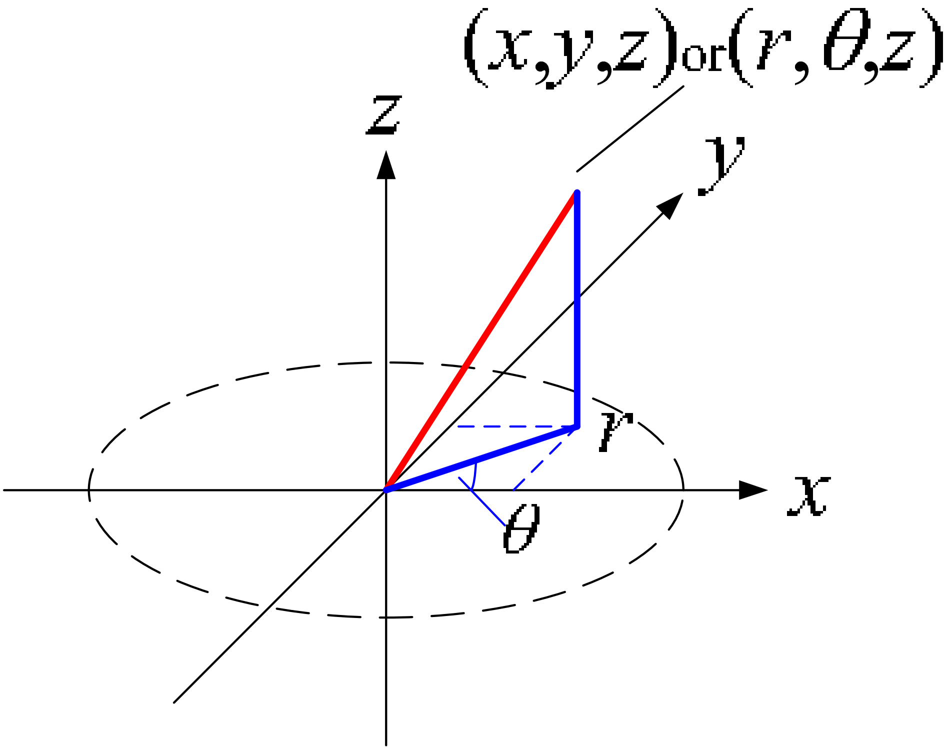
$$\frac{\partial}{\partial y} = (\sin\theta)\frac{\partial}{\partial r} + (\frac{\cos\theta}{r})\frac{\partial}{\partial\theta} + (0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = (0)\frac{\partial}{\partial r} + (0)\frac{\partial}{\partial\theta} + (1)\frac{\partial}{\partial z}$$

直交座標(x, y, z)⇒円筒座標(r, θ, z)

直交座標を円筒座標に変換する場合です。変数は下記で表されます。

$$\begin{cases} r = +\sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$



ベクトルの変換は次式となります。

$$\vec{\delta}_r = \vec{\delta}_x(\cos\theta) + \vec{\delta}_y(\sin\theta) + \vec{\delta}_z(0)$$

$$\vec{\delta}_\theta = \vec{\delta}_x(-\sin\theta) + \vec{\delta}_y(\cos\theta) + \vec{\delta}_z(0)$$

$$\vec{\delta}_z = \vec{\delta}_x(0) + \vec{\delta}_y(0) + \vec{\delta}_z(1)$$

作用素の変換は次式となります。

$$\frac{\partial}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y} + (0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \theta} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y} + (0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = (0)\frac{\partial}{\partial x} + (0)\frac{\partial}{\partial y} + (1)\frac{\partial}{\partial z}$$

また、

$$\frac{\partial}{\partial r}\vec{\delta}_r = 0, \quad \frac{\partial}{\partial r}\vec{\delta}_\theta = 0, \quad \frac{\partial}{\partial r}\vec{\delta}_z = 0$$

$$\frac{\partial}{\partial \theta}\vec{\delta}_r = \vec{\delta}_\theta, \quad \frac{\partial}{\partial \theta}\vec{\delta}_\theta = -\vec{\delta}_r, \quad \frac{\partial}{\partial \theta}\vec{\delta}_z = 0$$

$$\frac{\partial}{\partial z}\vec{\delta}_r = 0, \quad \frac{\partial}{\partial z}\vec{\delta}_\theta = 0, \quad \frac{\partial}{\partial z}\vec{\delta}_z = 0$$

従って、

$$\nabla = \vec{\delta}_r \frac{\partial}{\partial r} + \vec{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{\delta}_z \frac{\partial}{\partial z}$$

途中式

$$\frac{\partial}{\partial x} = (\cos\theta)\frac{\partial}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = (\sin\theta)\frac{\partial}{\partial r} + (\frac{\cos\theta}{r})\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial r} = \frac{1}{\cos\theta}\frac{\partial}{\partial x} + \frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial\theta} = (\frac{r}{\cos\theta})\frac{\partial}{\partial y} - (\frac{r\sin\theta}{\cos\theta})\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial\theta} = (\frac{r}{\cos\theta})\frac{\partial}{\partial y} - (\frac{r\sin\theta}{\cos\theta})\frac{1}{\cos\theta}\frac{\partial}{\partial x} - (\frac{r\sin\theta}{\cos\theta})\frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial\theta} + (\frac{\sin^2\theta}{\cos^2\theta})\frac{\partial}{\partial\theta} = (\frac{r}{\cos\theta})\frac{\partial}{\partial y} - (\frac{r\sin\theta}{\cos^2\theta})\frac{\partial}{\partial x}$$

$$(\cos^2\theta)\frac{\partial}{\partial\theta} + (\sin^2\theta)\frac{\partial}{\partial\theta} = (r\cos\theta)\frac{\partial}{\partial y} - (r\sin\theta)\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial\theta} = (r\cos\theta)\frac{\partial}{\partial y} - (r\sin\theta)\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} = (\cos\theta)\frac{\partial}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = (\sin\theta)\frac{\partial}{\partial r} + (\frac{\cos\theta}{r})\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial r} = \frac{1}{\cos\theta}\frac{\partial}{\partial x} + \frac{\sin\theta}{r\cos\theta}\{(r\cos\theta)\frac{\partial}{\partial y} - (r\sin\theta)\frac{\partial}{\partial x}\}$$

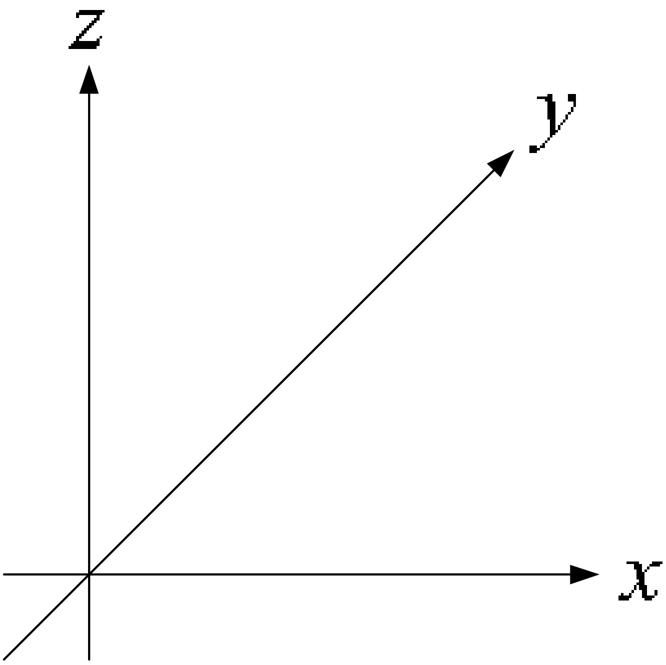
$$\frac{\partial}{\partial r} = \frac{1}{\cos\theta}\frac{\partial}{\partial x} + \frac{\sin\theta}{r\cos\theta}(r\cos\theta)\frac{\partial}{\partial y} - \frac{\sin\theta}{r\cos\theta}(r\sin\theta)\frac{\partial}{\partial x}$$

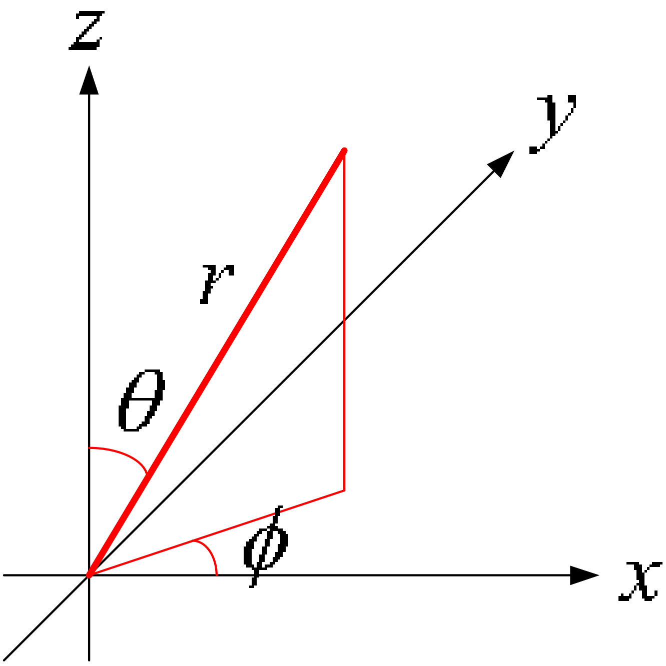
$$\frac{\partial}{\partial r} = \frac{1}{\cos\theta}\frac{\partial}{\partial x} - (\frac{\sin^2\theta}{\cos\theta})\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial r} = (\frac{1-\sin^2\theta}{\cos\theta})\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}$$

・球座標（極座標）
直交座標と球座標の座標変換について述べます。変数はそれぞれ下記が用いられます。
直交座標　(x, y, z)
球座標　(r, θ, ϕ)

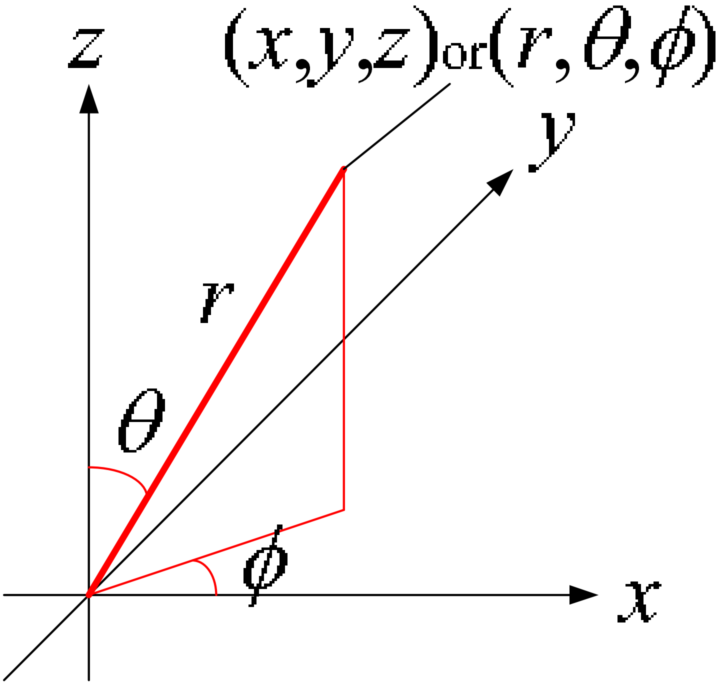




球座標(r, θ, ϕ)⇒直交座標(x, y, z)

球座標を直交座標に変換する場合です。変数は下記で表されます。

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



ベクトルの変換は次式となります。

$$\vec{\delta}_x = \vec{\delta}_r(\sin\theta\cos\phi) + \vec{\delta}_\theta(\cos\theta\cos\phi) + \vec{\delta}_\phi(-\sin\phi)$$

$$\vec{\delta}_y = \vec{\delta}_r(\sin\theta\sin\phi) + \vec{\delta}_\theta(\cos\theta\sin\phi) + \vec{\delta}_\phi(\cos\phi)$$

$$\vec{\delta}_z = \vec{\delta}_r(\cos\theta) + \vec{\delta}_\theta(-\sin\theta) + \vec{\delta}_\phi(0)$$

作用素の変換は次式となります。

$$\frac{\partial}{\partial x} = (\sin\theta\cos\phi)\frac{\partial}{\partial r} + (\frac{\cos\theta\cos\phi}{r})\frac{\partial}{\partial\theta} + (-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

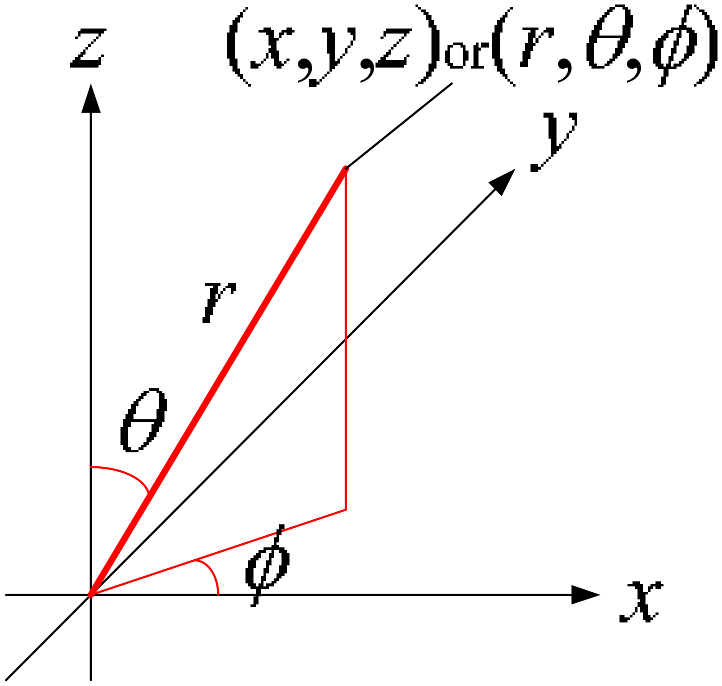
$$\frac{\partial}{\partial y} = (\sin\theta\sin\phi)\frac{\partial}{\partial r} + (\frac{\cos\theta\sin\phi}{r})\frac{\partial}{\partial\theta} + (\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = (\cos\theta)\frac{\partial}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial}{\partial\theta} + (0)\frac{\partial}{\partial\phi}$$

直交座標(x, y, z)⇒球座標(r, θ, ϕ)

直交座標を円筒座標に変換する場合です。変数は下記で表されます。

$$\begin{cases} r = +\sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2} / z) \\ \phi = \tan^{-1}(y / x) \end{cases}$$



ベクトルの変換は次式となります。

$$\vec{\delta}_r = \vec{\delta}_x(\sin\theta\cos\phi) + \vec{\delta}_y(\sin\theta\sin\phi) + \vec{\delta}_z(\cos\theta)$$

$$\vec{\delta}_\theta = \vec{\delta}_x(\cos\theta\cos\phi) + \vec{\delta}_y(\cos\theta\sin\phi) + \vec{\delta}_z(-\sin\theta)$$

$$\vec{\delta}_\phi = \vec{\delta}_x(-\sin\phi) + \vec{\delta}_y(\cos\phi) + \vec{\delta}_z(0)$$

作用素の変換は次式となります。

$$\frac{\partial}{\partial r}=\sin\theta\cos\phi\frac{\partial}{\partial x}+\sin\theta\sin\phi\frac{\partial}{\partial y}+\cos\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \theta}=r\cos\theta\cos\phi\frac{\partial}{\partial x}+r\cos\theta\sin\phi\frac{\partial}{\partial y}-r\sin\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \phi}=-r\sin\theta\sin\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\frac{\partial}{\partial y}+(0)\frac{\partial}{\partial z}$$

また、

$$\frac{\partial}{\partial r}\vec{\delta}_r=0,\quad \frac{\partial}{\partial r}\vec{\delta}_\theta=0,\quad \frac{\partial}{\partial r}\vec{\delta}_\phi=0$$

$$\frac{\partial}{\partial \theta}\vec{\delta}_r=\vec{\delta}_\theta,\quad \frac{\partial}{\partial \theta}\vec{\delta}_\theta=-\vec{\delta}_r,\quad \frac{\partial}{\partial \theta}\vec{\delta}_\phi=0$$

$$\frac{\partial}{\partial \phi}\vec{\delta}_r=\vec{\delta}_\phi(\sin\theta),\quad \frac{\partial}{\partial \phi}\vec{\delta}_\theta=\vec{\delta}_\phi(\cos\theta),\quad \frac{\partial}{\partial \phi}\vec{\delta}_\phi=-\vec{\delta}_r(\sin\theta)-\vec{\delta}_\theta(\cos\theta)$$

従って、

$$\nabla=\vec{\delta}_r\frac{\partial}{\partial r}+\vec{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\vec{\delta}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

途中式

$$\frac{\partial}{\partial x} = (\sin\theta\cos\phi)\frac{\partial}{\partial r} + (\frac{\cos\theta\cos\phi}{r})\frac{\partial}{\partial\theta} + (-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = (\sin\theta\sin\phi)\frac{\partial}{\partial r} + (\frac{\cos\theta\sin\phi}{r})\frac{\partial}{\partial\theta} + (\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = (\cos\theta)\frac{\partial}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial}{\partial\theta} + (0)\frac{\partial}{\partial\phi}$$

なので、

$$\frac{\partial}{\partial r} = \frac{1}{\cos\theta}\frac{\partial}{\partial z} + \frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}$$

を代入して

$$\frac{\partial}{\partial x} = (\sin\theta\cos\phi)(\frac{1}{\cos\theta}\frac{\partial}{\partial z} + \frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}) + (\frac{\cos\theta\cos\phi}{r})\frac{\partial}{\partial\theta}$$
$$+ (-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = (\sin\theta\sin\phi)(\frac{1}{\cos\theta}\frac{\partial}{\partial z} + \frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}) + (\frac{\cos\theta\sin\phi}{r})\frac{\partial}{\partial\theta}$$
$$+ (\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

整理して

$$\frac{\partial}{\partial x}=(\frac{\sin\theta\cos\phi}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin^2\theta\cos\phi}{r\cos\theta}\frac{\partial}{\partial\theta})+(\frac{\cos\theta\cos\phi}{r})\frac{\partial}{\partial\theta}$$
$$+(-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}=(\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin\theta^2\sin\phi}{r\cos\theta}\frac{\partial}{\partial\theta})+(\frac{\cos\theta\sin\phi}{r})\frac{\partial}{\partial\theta}$$
$$+(\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial x}-\frac{\sin\theta\cos\phi}{\cos\theta}\frac{\partial}{\partial z}=(\frac{\sin^2\theta\cos\phi}{r\cos\theta}+\frac{\cos\theta\cos\phi}{r})\frac{\partial}{\partial\theta}+(-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=(\frac{\sin\theta^2\sin\phi}{r\cos\theta}+\frac{\cos\theta\sin\phi}{r})\frac{\partial}{\partial\theta}+(\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial x}-\frac{\sin\theta\cos\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{1}{r}(\frac{\sin^2\theta\cos\phi+\cos^2\theta\cos\phi}{\cos\theta})\frac{\partial}{\partial\theta}+(-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{1}{r}(\frac{\sin\theta^2\sin\phi+\cos^2\theta\sin\phi}{\cos\theta})\frac{\partial}{\partial\theta}+(\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

従って、

$$\frac{\partial}{\partial x}-\frac{\sin\theta\cos\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{\cos\phi}{r\cos\theta}\frac{\partial}{\partial\theta}+(-\frac{\sin\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{\sin\phi}{r\cos\theta}\frac{\partial}{\partial\theta}+(\frac{\cos\phi}{r\sin\theta})\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta}=\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-\frac{r\cos\theta\sin\theta\cos\phi}{\cos\theta\cos\phi}\frac{\partial}{\partial z}+\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta}=\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\partial}{\partial z}+\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}$$

代入して、

$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{\sin\phi}{r\cos\theta}\left(\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\partial}{\partial z}+\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}\right)+\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{\sin\phi}{r\cos\theta}\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\sin\phi}{r\cos\theta}\frac{\partial}{\partial z}+\frac{\sin\phi}{r\cos\theta}\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}+\left(\frac{\cos\phi}{r\sin\theta}\right)\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}=\frac{\sin\phi}{\cos\phi}\frac{\partial}{\partial x}-\frac{\sin\theta\sin\phi}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin^2\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\phi}=-r\sin\theta\sin\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\frac{\partial}{\partial y}+(0)\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\theta}=\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\partial}{\partial z}+\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial\theta}=\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\partial}{\partial z}$$
$$+\frac{r\cos\theta\sin\phi}{r\sin\theta\cos\phi}(-r\sin\theta\sin\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial\theta}=\frac{r\cos\theta}{\cos\phi}\frac{\partial}{\partial x}-r\sin\theta\frac{\partial}{\partial z}-r\frac{\cos\theta\sin\phi}{\cos\phi}\sin\phi\frac{\partial}{\partial x}+r\cos\theta\sin\phi\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial\theta}=r\cos\theta\cos\phi\frac{\partial}{\partial x}+r\cos\theta\sin\phi\frac{\partial}{\partial y}-r\sin\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r}=\frac{1}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin\theta}{r\cos\theta}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial r}=\frac{1}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin\theta}{r\cos\theta}(r\cos\theta\cos\phi\frac{\partial}{\partial x}+r\cos\theta\sin\phi\frac{\partial}{\partial y}-r\sin\theta\frac{\partial}{\partial z})$$

$$\frac{\partial}{\partial r}=\frac{1}{\cos\theta}\frac{\partial}{\partial z}+\frac{\sin\theta}{r\cos\theta}r\cos\theta\cos\phi\frac{\partial}{\partial x}+\frac{\sin\theta}{r\cos\theta}r\cos\theta\sin\phi\frac{\partial}{\partial y}$$
$$-\frac{\sin\theta}{r\cos\theta}r\sin\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r} = +\sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + (\frac{1-\sin^2\theta}{\cos\theta})\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$$