・有限要素法

有限要素法 (Finite Element Method)とは、解析する領域を 3 角形や 4 角形などで細分化し、それら１つ１つに方程式を当てはめて、まとめて解くことで物理量を計算する方法です。

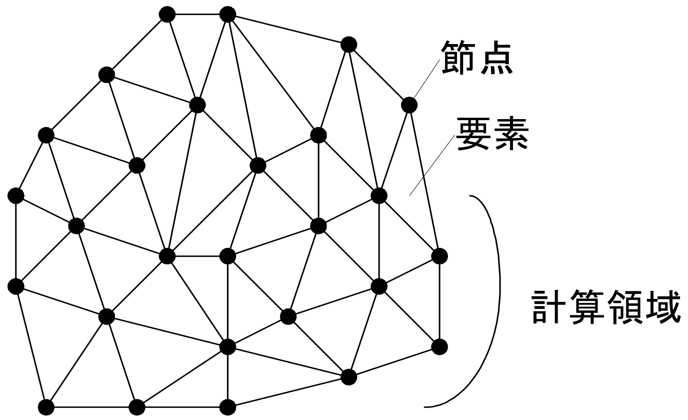
数値計算の目的は、導出された支配方程式をプログラム化して、現象をパソコンの画面上に可視化することで知見を得ることです。有限要素法もその手法の１つです。

流体力学で導出された微分方程式はそのままではプログラム化することはできません。有限要素法では、離散化（細分化すること）という作業が行われ、微分方程式が行列式に変換されることで、プログラムが可能になります。

有限要素法では、節点といくつかの節点から構成される要素が用いられます。各節点は、速度、座標などの情報を持っており、節点の物理量から各要素の積分値が計算されます。計算では、各要素で収支式の積分が行われます。

積分値は行列式で表されます。全要素の行列式を組み合わせ、１つの大きな多元１次連立方程式を解くことで、速度などの変数が計算されます。

この行列式の計算は、タイムステップごとに計算されます。タイムステップとは、総計算時間に対するユーザーが設定する微小時間です。つまり、時間のかかる地道な計算であり、現象は少しずつ解析されます。計算領域は、要素全体で表されます。

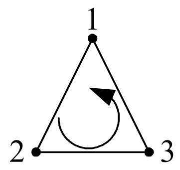


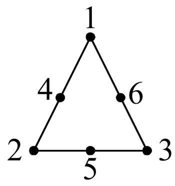
・節点、要素

要素にはいくつかの種類がります。

3 角形要素

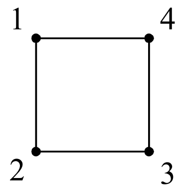
1 次要素　　　　2 次要素

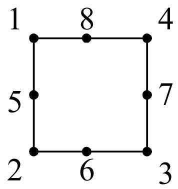




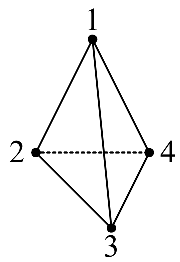
4 角形要素

1 次要素　　　　2 次要素

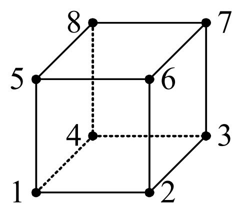




4 面体要素



6 面体要素

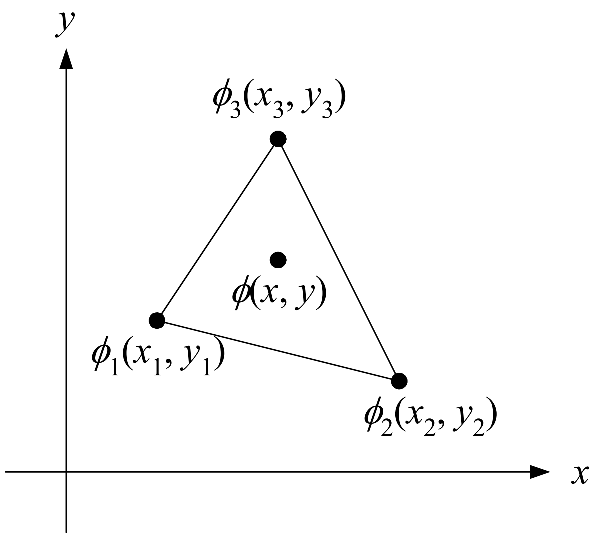


上図の様に各要素の節点には番号が割り振られます。割り振る順番は、反時計回りです。計算時には全節点に重複しない番号が割り振られますが、その番号とは異なります。

・３角形要素

要素内の任意の点における物理量の計算には形状関数が使用されます。形状関数は、要素ごとに異なります。また、各要素には 1 次要素、2 次要素などの高次要素もあります。ここでは、最も簡単な 3 角形 1 次要素の形状関数を導出します。

・3 角形 1 次要素



上図の様に xy 座標に 1 つの 3 角形要素を考えます。1 次要素の場合は、物理量ϕは次式となります。

$$\phi(x, y) = \alpha + \beta x + \gamma y$$

ここで、α, β, γは未知数です。これら 3 つの未知数を 3 つの節点の物理量から計算します。

$$\begin{cases} \phi_1 = \alpha + \beta x_1 + \gamma y_1 \\ \phi_2 = \alpha + \beta x_2 + \gamma y_2 \\ \phi_3 = \alpha + \beta x_3 + \gamma y_3 \end{cases}$$

これを計算すると次式となります。

$$\alpha = (x_2 y_3 - x_3 y_2)\phi_1 + (x_3 y_1 - x_1 y_3)\phi_2 + (x_1 y_2 - x_2 y_1)\phi_3$$
$$\beta = (y_2 - y_3)\phi_1 + (y_3 - y_1)\phi_2 + (y_1 - y_2)\phi_3$$
$$\gamma = (x_3 - x_2)\phi_1 + (x_1 - x_3)\phi_2 + (x_2 - x_1)\phi_3$$

これを物理量ϕの式に代入すると次式が得られます。

$$\begin{aligned} \phi(x, y) &= N_1\phi_1 + N_2\phi_2 + N_3\phi_3 \\ &= L_1\phi_1 + L_2\phi_2 + L_3\phi_3 \end{aligned}$$

ここで、L_1, L_2, L_3[-]は形状関数と呼ばれ下図を用いて定義できます。

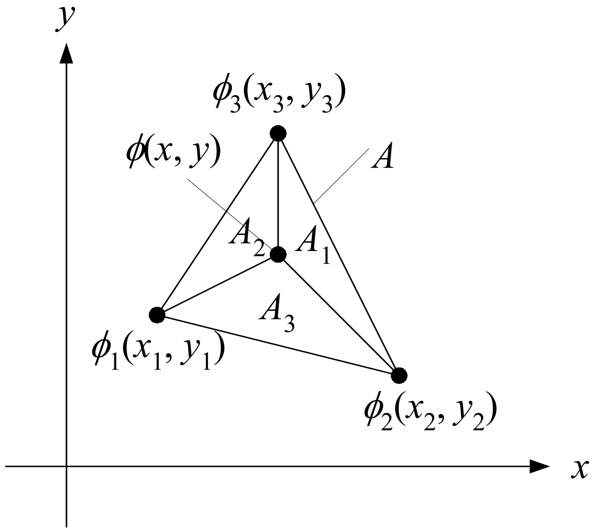
$$L_1 = \frac{A_1}{A}, L_2 = \frac{A_2}{A}, L_3 = \frac{A_3}{A}$$

また、

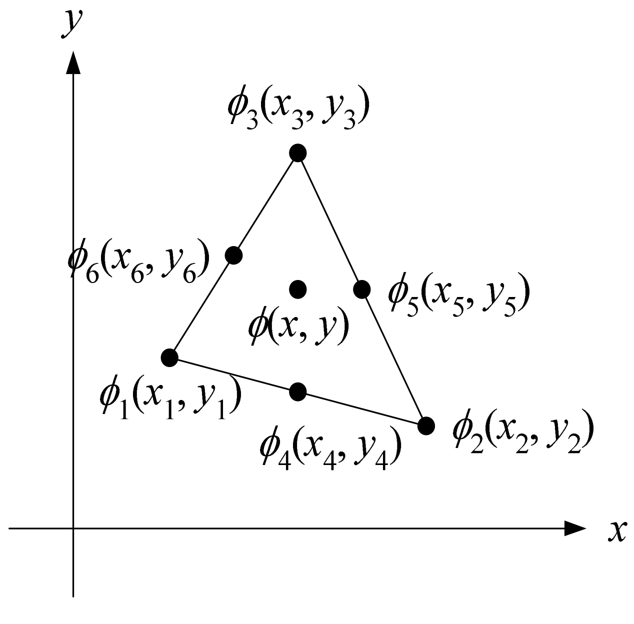
$$N_1 = L_1, \quad N_2 = L_2, \quad N_3 = L_3$$

$$N_1 + N_2 + N_3 = 1$$

従って、3 角形要素内の任意の位置の物理量は、内挿関数 N_i と節点の物理量 ϕ_i で表されます。



・3 角形 2 次要素



次に 3 角形 2 次要素の場合は、物理量 ϕ は次式となります。

$$\phi(x, y) = \alpha + \beta x + \gamma y + \xi x^2 + \eta xy + \zeta y^2$$

未知数は 6 個で、要素の 6 個の節点から 6 元 1 次方程式を解くことになります。

$$
\begin{cases}
\phi_1 = \alpha + \beta x_1 + \gamma y_1 + \xi x_1^2 + \eta x_1 y_1 + \zeta y_1^2 \\
\phi_2 = \alpha + \beta x_2 + \gamma y_2 + \xi x_2^2 + \eta x_2 y_2 + \zeta y_2^2 \\
\phi_3 = \alpha + \beta x_3 + \gamma y_3 + \xi x_3^2 + \eta x_3 y_3 + \zeta y_3^2 \\
\phi_4 = \alpha + \beta x_4 + \gamma y_4 + \xi x_4^2 + \eta x_4 y_4 + \zeta y_4^2 \\
\phi_5 = \alpha + \beta x_5 + \gamma y_5 + \xi x_5^2 + \eta x_5 y_5 + \zeta y_5^2 \\
\phi_6 = \alpha + \beta x_6 + \gamma y_6 + \xi x_6^2 + \eta x_6 y_6 + \zeta y_6^2
\end{cases}
$$

これを解くと次式が得られます。

$$
\begin{aligned}
\phi(x, y) &= N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 + N_5\phi_5 + N_6\phi_6 \\
&= L_1(2L_1 - 1)\phi_1 + L_2(2L_2 - 1)\phi_2 + L_3(2L_3 - 1)\phi_3 \\
&\quad + 4L_1L_2\phi_4 + 4L_2L_3\phi_5 + 4L_3L_1\phi_6
\end{aligned}
$$

ここで、

$$
\begin{aligned}
N_1 &= L_1(2L_1 - 1) \\
N_2 &= L_2(2L_2 - 1) \\
N_3 &= L_3(2L_3 - 1) \\
N_4 &= 4L_1L_2 \\
N_5 &= 4L_2L_3 \\
N_6 &= 4L_3L_1
\end{aligned}
$$

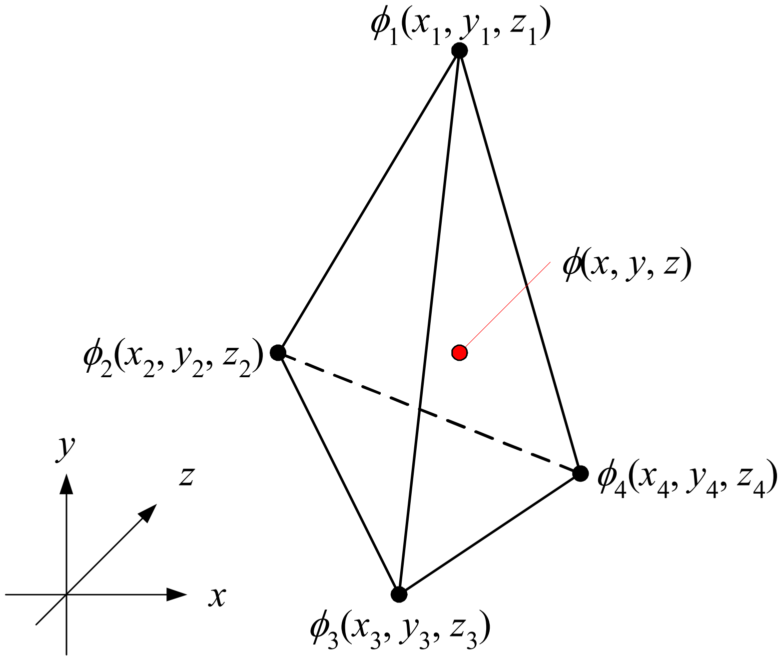
また、

$$
\sum_{i=1}^{6} N_i = 1
$$

従って、3 角形要素内の任意の位置の物理量は、内挿関数 N_i と節点の物理量 ϕ_i で表されます。

・4 面体 1 次要素の形状関数

4 面体 1 次要素の場合も同様に定義されます。



上図の様に xyz 座標に 1 つの 4 面体要素を考える。1 次要素の場合は、物理量ϕは次式となります。

$$\phi(x, y, z) = \alpha + \beta x + \gamma y + \xi z$$

ここで、$\alpha, \beta, \gamma, \eta$は未知数です。これら 4 つの未知数を 4 つの節点の物理量から計算します。

$$\begin{cases} \phi_1 = \alpha + \beta x_1 + \gamma y_1 + \xi z_1 \\ \phi_2 = \alpha + \beta x_2 + \gamma y_2 + \xi z_2 \\ \phi_3 = \alpha + \beta x_3 + \gamma y_3 + \xi z_3 \\ \phi_4 = \alpha + \beta x_4 + \gamma y_4 + \xi z_4 \end{cases}$$

この連立方程式の解を物理量ϕの式に代入すると次式が得られます。

$$\begin{aligned} \phi(x, y) &= N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 \\ &= L_1\phi_1 + L_2\phi_2 + L_3\phi_3 + L_4\phi_4 \end{aligned}$$

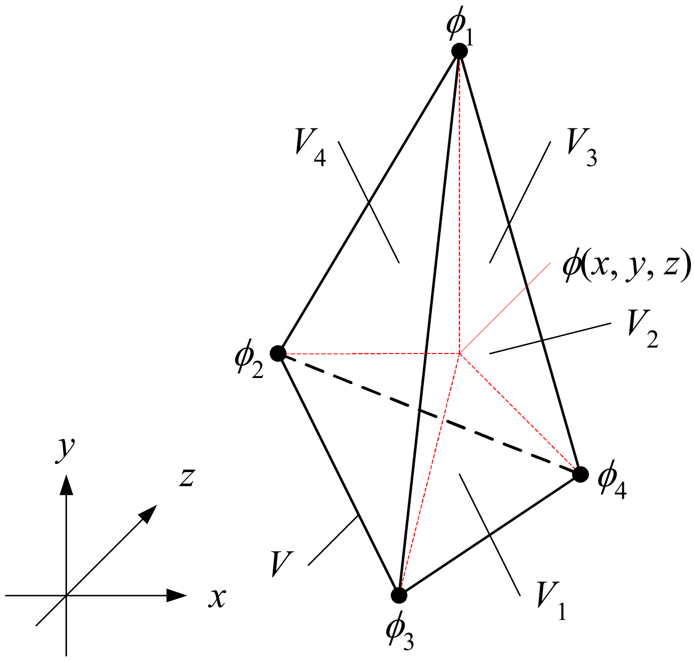
ここで、L_1, L_2, L_3[-]は形状関数と呼ばれ下図を用いて定義できます。

$$L_1 = \frac{V_1}{V}, L_2 = \frac{V_2}{V}, L_3 = \frac{V_3}{V}, L_4 = \frac{V_4}{V}$$

また、

$$N_1 = L_1, \quad N_2 = L_2, \quad N_3 = L_3, \quad N_4 = L_4$$

$$N_1 + N_2 + N_3 + N_4 = 1$$



・4 面体要素の内挿関数

導出した 4 面体要素の形状関数を用いて内挿関数 N_i を次式で定義します。

4 面体 1 次要素

$$
\begin{aligned}
\phi &= N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 \\
&= L_1\phi_1 + L_2\phi_2 + L_3\phi_3 + L_4\phi_4
\end{aligned}
$$

また、

$$N_1 + N_2 + N_3 + N_4 = 1$$

従って、4 面体要素内の任意の位置の物理量は、内挿関数 N_i と節点の物理量 $\phi_1, \phi_2, \phi_3, \phi_4$ で表されます。

・要素の積分

要素の積分には下記の公式が用いられます。

3 角形要素

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!} 2S$$

ここで、S は要素の面積です。

4 面体要素

$$\int_V L_1^p L_2^q L_3^r L_4^s dV = \frac{p!q!r!s!}{(p+q+r+s+3)!} 6V$$

ここで、V は要素の体積です。

3 角形要素の面積は次式となります。

$$S = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

行列式は下記で計算されます。

$$\det A = \sum_{\sigma \in \sigma_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$\varepsilon(\sigma)$を$\sigma \in \sigma_n$の符号とすると

(1)$\varepsilon(\tau\sigma) = \varepsilon(\sigma)\varepsilon(\tau)$

(2)$\varepsilon(1) = 1$

(3)$\varepsilon(\sigma^{-1}) = \varepsilon(\sigma)$

(4)ρが互換であれば$\varepsilon(\rho) = -1$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \sum_{\sigma\in\sigma_3} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$$

$$= \sum_{\sigma\in\sigma_3} \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$$

$$= \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} a_{11}a_{22}a_{33} + \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} a_{11}a_{23}a_{32}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} a_{12}a_{21}a_{33} + \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} a_{12}a_{23}a_{31}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} a_{13}a_{21}a_{32} + \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} a_{13}a_{22}a_{31}$$

$$= \varepsilon\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} a_{11}a_{22}a_{33} \qquad + \varepsilon\left((2\ 3)\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right) a_{11}a_{23}a_{32}$$

$$+ \varepsilon\left((1\ 2)\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right) a_{12}a_{21}a_{33} \qquad + \varepsilon\left((1\ 2)(2\ 3)\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right) a_{12}a_{23}a_{31}$$

$$+ \varepsilon\left((1\ 3)(2\ 3)\right)\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} a_{13}a_{21}a_{32} + \varepsilon\left((1\ 3)\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right) a_{13}a_{22}a_{31}$$

$$= (-1)^0 a_{11}a_{22}a_{33} + (-1)^1 a_{11}a_{23}a_{32}$$

$$+ (-1)^1 a_{12}a_{21}a_{33} + (-1)^2 a_{12}a_{23}a_{31}$$

$$+ (-1)^2 a_{13}a_{21}a_{32} + (-1)^1 a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

従って、3 角形の面積 S は、

$$S = \frac{1}{2}\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$= \frac{1}{2}(x_2 y_3 - y_2 x_3 - x_1 y_3 + x_1 y_2 + y_1 x_3 - y_1 x_2)$$

$$a_{11} = 1, a_{12} = x_1, a_{13} = y_1$$

$$a_{21} = 1, a_{22} = x_2, a_{23} = y_2$$

$$a_{31} = 1, a_{32} = x_3, a_{33} = y_3$$

4 面体要素の体積は次式となります。

$$V = \frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}$$

行列式は下記で計算されます。

$$\det A = \sum_{\sigma \in \sigma_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$\varepsilon(\sigma)$を$\sigma \in \sigma_n$の符号とすると

(1)$\varepsilon(\tau\sigma) = \varepsilon(\sigma)\varepsilon(\tau)$

(2)$\varepsilon(1) = 1$

(3)$\varepsilon(\sigma^{-1}) = \varepsilon(\sigma)$

(4)ρが互換であれば$\varepsilon(\rho) = -1$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{43} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \sum_{\sigma\in\sigma_4} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} a_{4\sigma(4)}$$

$$= \sum_{\sigma\in\sigma_4} \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} a_{4\sigma(4)}$$

$$= \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} a_{11}a_{22}a_{33}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} a_{11}a_{22}a_{34}a_{43}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} a_{11}a_{23}a_{32}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} a_{11}a_{23}a_{34}a_{42}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} a_{11}a_{24}a_{32}a_{43} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} a_{11}a_{24}a_{33}a_{42}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} a_{12}a_{21}a_{33}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} a_{12}a_{21}a_{34}a_{43}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} a_{12}a_{23}a_{31}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} a_{12}a_{23}a_{34}a_{41}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} a_{12}a_{24}a_{31}a_{43} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} a_{12}a_{24}a_{33}a_{41}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} a_{13}a_{21}a_{32}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} a_{13}a_{21}a_{34}a_{42}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} a_{13}a_{22}a_{31}a_{44} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} a_{13}a_{22}a_{34}a_{41}$$

$$+ \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} a_{13}a_{24}a_{31}a_{42} + \varepsilon\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} a_{13}a_{24}a_{32}a_{41}$$

$$= \varepsilon\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}a_{11}a_{22}a_{33}a_{44} + \varepsilon\left((3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{11}a_{22}a_{34}a_{43}$$

$$+ \varepsilon\left((2\,3)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{11}a_{23}a_{32}a_{44} + \varepsilon\left((2\,3)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{11}a_{23}a_{34}a_{42}$$

$$+ \varepsilon\left((2\,4)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{11}a_{24}a_{32}a_{43} + \varepsilon\left((2\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{11}a_{24}a_{33}a_{42}$$

$$+ \varepsilon\left((1\,2)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{21}a_{33}a_{44} + \varepsilon\left((1\,2)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{21}a_{34}a_{43}$$

$$+ \varepsilon\left((1\,2)(2\,3)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{23}a_{31}a_{44} + \varepsilon\left((1\,2)(2\,3)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{23}a_{34}a_{41}$$

$$+ \varepsilon\left((1\,2)(2\,4)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{24}a_{31}a_{43} + \varepsilon\left((1\,2)(2\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{12}a_{24}a_{33}a_{41}$$

$$+ \varepsilon\left((1\,3)(2\,3)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{21}a_{32}a_{44} + \varepsilon\left((1\,3)(2\,3)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{21}a_{34}a_{42}$$

$$+ \varepsilon\left((1\,3)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{22}a_{31}a_{44} + \varepsilon\left((1\,3)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{22}a_{34}a_{41}$$

$$+ \varepsilon\left((1\,3)(2\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{24}a_{31}a_{42} + \varepsilon\left((1\,3)(2\,4)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{13}a_{24}a_{32}a_{41}$$

$$+ \varepsilon\left((1\,4)(2\,4)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{21}a_{32}a_{43} + \varepsilon\left((1\,4)(2\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{21}a_{33}a_{42}$$

$$+ \varepsilon\left((1\,4)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{22}a_{31}a_{43} + \varepsilon\left((1\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{22}a_{33}a_{41}$$

$$+ \varepsilon\left((1\,4)(2\,3)(3\,4)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{23}a_{31}a_{42} + \varepsilon\left((1\,4)(2\,3)\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\right)a_{14}a_{23}a_{32}a_{41}$$

$$
\begin{aligned}
&= (-1)^0 a_{11}a_{22}a_{33}a_{44} + (-1)^1 a_{11}a_{22}a_{34}a_{43} + (-1)^1 a_{11}a_{23}a_{32}a_{44} \\
&+ (-1)^2 a_{11}a_{23}a_{34}a_{42} + (-1)^2 a_{11}a_{24}a_{32}a_{43} + (-1)^1 a_{11}a_{24}a_{33}a_{42} \\
&+ (-1)^1 a_{12}a_{21}a_{33}a_{44} + (-1)^2 a_{12}a_{21}a_{34}a_{43} + (-1)^2 a_{12}a_{23}a_{31}a_{44} \\
&+ (-1)^3 a_{12}a_{23}a_{34}a_{41} + (-1)^3 a_{12}a_{24}a_{31}a_{43} + (-1)^2 a_{12}a_{24}a_{33}a_{41} \\
&+ (-1)^2 a_{13}a_{21}a_{32}a_{44} + (-1)^3 a_{13}a_{21}a_{34}a_{42} + (-1)^1 a_{13}a_{22}a_{31}a_{44} \\
&+ (-1)^2 a_{13}a_{22}a_{34}a_{41} + (-1)^2 a_{13}a_{24}a_{31}a_{42} + (-1)^3 a_{13}a_{24}a_{32}a_{41} \\
&+ (-1)^3 a_{14}a_{21}a_{32}a_{43} + (-1)^2 a_{14}a_{21}a_{33}a_{42} + (-1)^2 a_{14}a_{22}a_{31}a_{43} \\
&+ (-1)^1 a_{14}a_{22}a_{33}a_{41} + (-1)^3 a_{14}a_{23}a_{31}a_{42} + (-1)^2 a_{14}a_{23}a_{32}a_{41}
\end{aligned}
$$

$$
\begin{aligned}
&= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} \\
&+ a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} \\
&- a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} \\
&- a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{24}a_{33}a_{41} \\
&+ a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} \\
&+ a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} \\
&- a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} \\
&- a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41}
\end{aligned}
$$

従って、４面体の体積Vは、

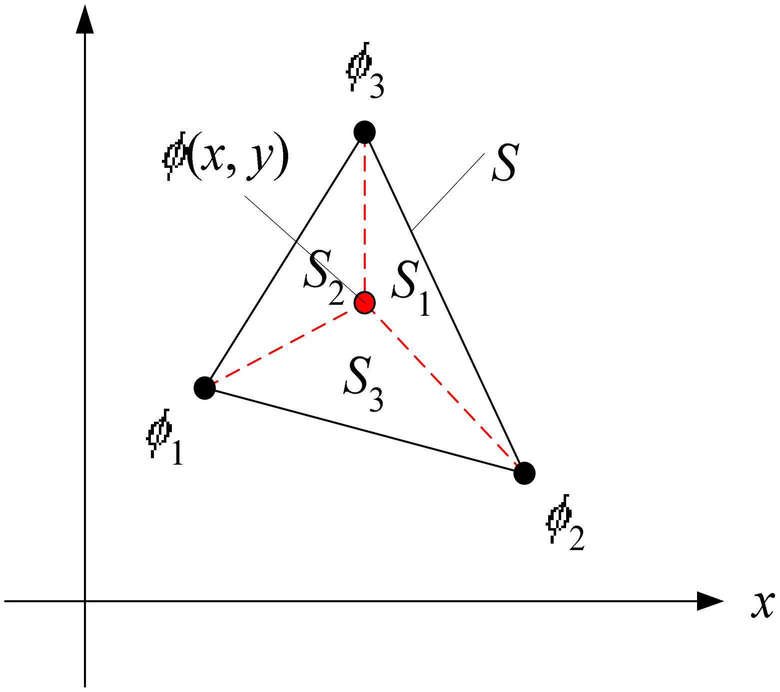
$$V=\frac{1}{6}\det\begin{pmatrix}1 & x_1 & y_1 & z_1\\ 1 & x_2 & y_2 & z_2\\ 1 & x_3 & y_3 & z_3\\ 1 & x_4 & y_4 & z_4\end{pmatrix}$$

$$=\frac{1}{6}(x_2y_3z_4-x_2z_3y_4-y_2x_3z_4+y_2z_3x_4+z_2x_3y_4-z_2y_3x_4$$
$$-x_1y_3z_4+x_1z_3y_4+x_1y_2z_4-x_1y_2z_3-x_1z_2y_4+x_1z_2y_3$$
$$+y_1x_3z_4-y_1z_3x_4-y_1x_2z_4+y_1x_2z_3+y_1z_2x_4-y_1z_2x_3$$
$$-z_1x_3y_4+z_1y_3x_4+z_1x_2y_4-z_1x_2y_3-z_1y_2x_4+z_1y_2x_3)$$

ここで、

$$a_{11}=1,a_{12}=x_1,a_{13}=y_1,a_{14}=z_1$$
$$a_{21}=1,a_{22}=x_2,a_{23}=y_2,a_{24}=z_2$$
$$a_{31}=1,a_{32}=x_3,a_{33}=y_3,a_{34}=z_3$$
$$a_{41}=1,a_{42}=x_4,a_{43}=y_4,a_{44}=z_4$$

・内挿関数の微分



3 角形の面積 S

$$S = \frac{1}{2}\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$= \frac{1}{2}(x_2 y_3 - y_2 x_3 - x_1 y_3 + x_1 y_2 + y_1 x_3 - y_1 x_2)$$

ここで、

$$a_{11} = 1, a_{12} = x_1, a_{13} = y_1$$
$$a_{21} = 1, a_{22} = x_2, a_{23} = y_2$$
$$a_{31} = 1, a_{32} = x_3, a_{33} = y_3$$

同様に、

$$S_1 = \frac{1}{2}\det\begin{pmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$= \frac{1}{2}(x_2 y_3 - y_2 x_3 - x y_3 + x y_2 + y x_3 - y x_2)$$

$$S_2 = \frac{1}{2}\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$= \frac{1}{2}(xy_3 - yx_3 - x_1y_3 + x_1y + y_1x_3 - y_1x)$$

$$S_3 = \frac{1}{2}\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{pmatrix}$$

$$= \frac{1}{2}(x_2y - y_2x - x_1y + x_1y_2 + y_1x - y_1x_2)$$

形状関数 N_1 の微分

$$\frac{\partial N_1}{\partial x} = \frac{\partial L_1}{\partial x} = \frac{\partial (S_1/S)}{\partial x} = \frac{1}{S}\frac{\partial S_1}{\partial x}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{2}(x_2y_3 - y_2x_3 - xy_3 + xy_2 + yx_3 - yx_2)$$

$$= \frac{1}{2S}(-y_3 + y_2)$$

$$\frac{\partial N_1}{\partial y} = \frac{\partial L_1}{\partial y} = \frac{\partial (S_1/S)}{\partial y} = \frac{1}{S}\frac{\partial S_1}{\partial y}$$

$$= \frac{1}{S}\frac{\partial}{\partial y}\frac{1}{2}(x_2y_3 - y_2x_3 - xy_3 + xy_2 + yx_3 - yx_2)$$

$$= \frac{1}{2S}(x_3 - x_2)$$

形状関数 N_2 の微分

$$\frac{\partial N_2}{\partial x} = \frac{\partial L_2}{\partial x} = \frac{\partial (S_2/S)}{\partial x} = \frac{1}{S}\frac{\partial S_2}{\partial x}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{2}(xy_3 - yx_3 - x_1y_3 + x_1y + y_1x_3 - y_1x)$$

$$= \frac{1}{2S}(y_3 - y_1)$$

$$\frac{\partial N_2}{\partial y} = \frac{\partial L_2}{\partial y} = \frac{\partial (S_2 / S)}{\partial y} = \frac{1}{S}\frac{\partial S_2}{\partial y}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{2}(xy_3 - yx_3 - x_1 y_3 + x_1 y + y_1 x_3 - y_1 x)$$

$$= \frac{1}{2S}(-x_3 + x_1)$$

形状関数 N_3 の微分

$$\frac{\partial N_3}{\partial x} = \frac{\partial L_3}{\partial x} = \frac{\partial (S_3 / S)}{\partial x} = \frac{1}{S}\frac{\partial S_3}{\partial x}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{2}(x_2 y - y_2 x - x_1 y + x_1 y_2 + y_1 x - y_1 x_2)$$

$$= \frac{1}{2S}(-y_2 + y_1)$$

$$\frac{\partial N_3}{\partial y} = \frac{\partial L_3}{\partial y} = \frac{\partial (S_3 / S)}{\partial y} = \frac{1}{S}\frac{\partial S_3}{\partial y}$$

$$= \frac{1}{S}\frac{\partial}{\partial x}\frac{1}{2}(x_2 y - y_2 x - x_1 y + x_1 y_2 + y_1 x - y_1 x_2)$$

$$= \frac{1}{2S}(x_2 - x_1)$$

形状関数の微分まとめ

$$c_{1x} = \frac{\partial N_1}{\partial x} = \frac{\partial L_1}{\partial x} = \frac{1}{2S}(-y_3 + y_2)$$

$$c_{1y} = \frac{\partial N_1}{\partial y} = \frac{\partial L_1}{\partial y} = \frac{1}{2S}(x_3 - x_2)$$

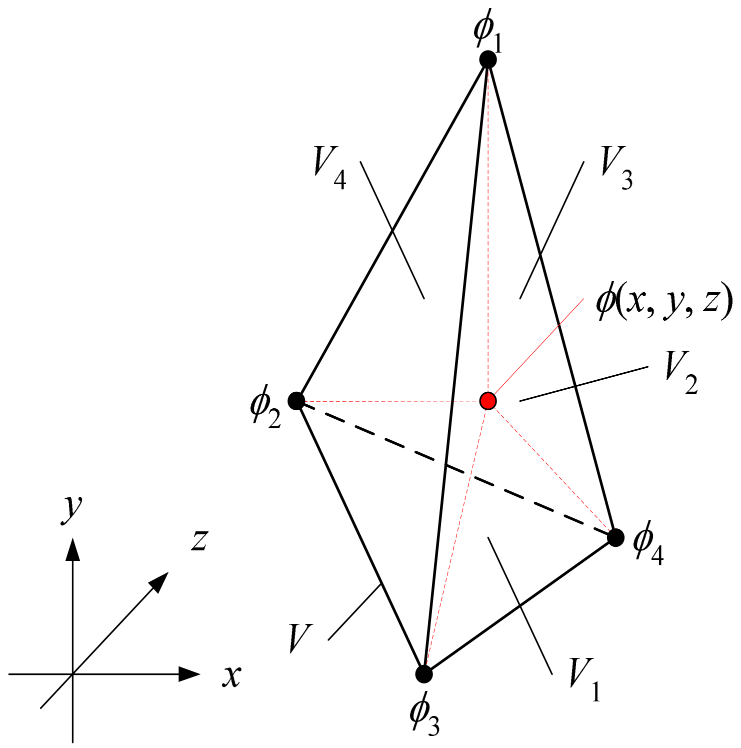
$$c_{2x} = \frac{\partial N_2}{\partial x} = \frac{\partial L_2}{\partial x} = \frac{1}{2S}(y_3 - y_1)$$

$$c_{2y} = \frac{\partial N_2}{\partial y} = \frac{\partial L_2}{\partial y} = \frac{1}{2S}(-x_3 + x_1)$$

$$c_{3x} = \frac{\partial N_3}{\partial x} = \frac{\partial L_3}{\partial x} = \frac{1}{2S}(-y_2 + y_1)$$

$$c_{3y} = \frac{\partial N_3}{\partial y} = \frac{\partial L_3}{\partial y} = \frac{1}{2S}(x_2 - x_1)$$

・内挿関数の微分



4 面体の体積 V

$$V = \frac{1}{6}\det\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}$$

$$= \frac{1}{6}(x_2 y_3 z_4 - x_2 z_3 y_4 - y_2 x_3 z_4 + y_2 z_3 x_4 + z_2 x_3 y_4 - z_2 y_3 x_4$$
$$- x_1 y_3 z_4 + x_1 z_3 y_4 + x_1 y_2 z_4 - x_1 y_2 z_3 - x_1 z_2 y_4 + x_1 z_2 y_3$$
$$+ y_1 x_3 z_4 - y_1 z_3 x_4 - y_1 x_2 z_4 + y_1 x_2 z_3 + y_1 z_2 x_4 - y_1 z_2 x_3$$
$$- z_1 x_3 y_4 + z_1 y_3 x_4 + z_1 x_2 y_4 - z_1 x_2 y_3 - z_1 y_2 x_4 + z_1 y_2 x_3)$$

ここで、

$$a_{11} = 1, a_{12} = x_1, a_{13} = y_1, a_{14} = z_1$$
$$a_{21} = 1, a_{22} = x_2, a_{23} = y_2, a_{24} = z_2$$
$$a_{31} = 1, a_{32} = x_3, a_{33} = y_3, a_{34} = z_3$$
$$a_{41} = 1, a_{42} = x_4, a_{43} = y_4, a_{44} = z_4$$

同様に、

$$V_1 = \frac{1}{6}\det\begin{pmatrix} 1 & x & y & z \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}$$

$$= \frac{1}{6}(x_2 y_3 z_4 - x_2 z_3 y_4 - y_2 x_3 z_4 + y_2 z_3 x_4 + z_2 x_3 y_4 - z_2 y_3 x_4$$

$$- x y_3 z_4 + x z_3 y_4 + x y_2 z_4 - x y_2 z_3 - x z_2 y_4 + x z_2 y_3$$

$$+ y x_3 z_4 - y z_3 x_4 - y x_2 z_4 + y x_2 z_3 + y z_2 x_4 - y z_2 x_3$$

$$- z x_3 y_4 + z y_3 x_4 + z x_2 y_4 - z x_2 y_3 - z y_2 x_4 + z y_2 x_3)$$

$$V_2 = \frac{1}{6}\det\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x & y & z \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}$$

$$= \frac{1}{6}(x y_3 z_4 - x z_3 y_4 - y x_3 z_4 + y z_3 x_4 + z x_3 y_4 - z y_3 x_4$$

$$- x_1 y_3 z_4 + x_1 z_3 y_4 + x_1 y z_4 - x_1 y z_3 - x_1 z y_4 + x_1 z y_3$$

$$+ y_1 x_3 z_4 - y_1 z_3 x_4 - y_1 x z_4 + y_1 x z_3 + y_1 z x_4 - y_1 z x_3$$

$$- z_1 x_3 y_4 + z_1 y_3 x_4 + z_1 x y_4 - z_1 x y_3 - z_1 y x_4 + z_1 y x_3)$$

$$V_3 = \frac{1}{6}\det\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x & y & z \\ 1 & x_4 & y_4 & z_4 \end{pmatrix}$$

$$= \frac{1}{6}(x_2 y z_4 - x_2 z y_4 - y_2 x z_4 + y_2 z x_4 + z_2 x y_4 - z_2 y x_4$$

$$- x_1 y z_4 + x_1 z y_4 + x_1 y_2 z_4 - x_1 y_2 z - x_1 z_2 y_4 + x_1 z_2 y$$

$$+ y_1 x z_4 - y_1 z x_4 - y_1 x_2 z_4 + y_1 x_2 z + y_1 z_2 x_4 - y_1 z_2 x$$

$$- z_1 x y_4 + z_1 y x_4 + z_1 x_2 y_4 - z_1 x_2 y - z_1 y_2 x_4 + z_1 y_2 x)$$

$$V_4 = \frac{1}{6}\det\begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x & y & z \end{pmatrix}$$

$$= \frac{1}{6}(x_2 y_3 z - x_2 z_3 y - y_2 x_3 z + y_2 z_3 x + z_2 x_3 y - z_2 y_3 x$$
$$- x_1 y_3 z + x_1 z_3 y + x_1 y_2 z - x_1 y_2 z_3 - x_1 z_2 y + x_1 z_2 y_3$$
$$+ y_1 x_3 z - y_1 z_3 x - y_1 x_2 z + y_1 x_2 z_3 + y_1 z_2 x - y_1 z_2 x_3$$
$$- z_1 x_3 y + z_1 y_3 x + z_1 x_2 y - z_1 x_2 y_3 - z_1 y_2 x + z_1 y_2 x_3)$$

1 次の形状関数

$$N_1 = L_1 = \frac{V_1}{V},$$
$$N_2 = L_2 = \frac{V_2}{V},$$
$$N_3 = L_3 = \frac{V_3}{V},$$
$$N_4 = L_4 = \frac{V_4}{V}$$

形状関数 N_1 の微分

$$\frac{\partial N_1}{\partial x} = \frac{\partial L_1}{\partial x} = \frac{\partial (V_1 / V)}{\partial x} = \frac{1}{V}\frac{\partial V_1}{\partial x}$$

$$= \frac{1}{V}\frac{\partial}{\partial x}\frac{1}{6}(x_2 y_3 z_4 - x_2 z_3 y_4 - y_2 x_3 z_4 + y_2 z_3 x_4 + z_2 x_3 y_4 - z_2 y_3 x_4$$
$$- x y_3 z_4 + x z_3 y_4 + x y_2 z_4 - x y_2 z_3 - x z_2 y_4 + x z_2 y_3$$
$$+ y x_3 z_4 - y z_3 x_4 - y x_2 z_4 + y x_2 z_3 + y z_2 x_4 - y z_2 x_3$$
$$- z x_3 y_4 + z y_3 x_4 + z x_2 y_4 - z x_2 y_3 - z y_2 x_4 + z y_2 x_3)$$

$$= \frac{1}{6V}(-y_3 z_4 + z_3 y_4 + y_2 z_4 - y_2 z_3 - z_2 y_4 + z_2 y_3)$$

$$\frac{\partial N_1}{\partial y} = \frac{\partial L_1}{\partial y} = \frac{\partial (V_1 / V)}{\partial y} = \frac{1}{V} \frac{\partial V_1}{\partial y}$$

$$= \frac{1}{V} \frac{\partial}{\partial y} \frac{1}{6} (x_2 y_3 z_4 - x_2 z_3 y_4 - y_2 x_3 z_4 + y_2 z_3 x_4 + z_2 x_3 y_4 - z_2 y_3 x_4$$

$$- x y_3 z_4 + x z_3 y_4 + x y_2 z_4 - x y_2 z_3 - x z_2 y_4 + x z_2 y_3$$

$$+ y x_3 z_4 - y z_3 x_4 - y x_2 z_4 + y x_2 z_3 + y z_2 x_4 - y z_2 x_3$$

$$- z x_3 y_4 + z y_3 x_4 + z x_2 y_4 - z x_2 y_3 - z y_2 x_4 + z y_2 x_3)$$

$$= \frac{1}{6V} (x_3 z_4 - z_3 x_4 - x_2 z_4 + x_2 z_3 + z_2 x_4 - z_2 x_3)$$

$$\frac{\partial N_1}{\partial z} = \frac{\partial L_1}{\partial z} = \frac{\partial (V_1 / V)}{\partial z} = \frac{1}{V} \frac{\partial V_1}{\partial z}$$

$$= \frac{1}{V} \frac{\partial}{\partial z} \frac{1}{6} (x_2 y_3 z_4 - x_2 z_3 y_4 - y_2 x_3 z_4 + y_2 z_3 x_4 + z_2 x_3 y_4 - z_2 y_3 x_4$$

$$- x y_3 z_4 + x z_3 y_4 + x y_2 z_4 - x y_2 z_3 - x z_2 y_4 + x z_2 y_3$$

$$+ y x_3 z_4 - y z_3 x_4 - y x_2 z_4 + y x_2 z_3 + y z_2 x_4 - y z_2 x_3$$

$$- z x_3 y_4 + z y_3 x_4 + z x_2 y_4 - z x_2 y_3 - z y_2 x_4 + z y_2 x_3)$$

$$= \frac{1}{6V} (-x_3 y_4 + y_3 x_4 + x_2 y_4 - x_2 y_3 - y_2 x_4 + y_2 x_3)$$

形状関数 N_2 の微分

$$\frac{\partial N_2}{\partial x} = \frac{\partial L_2}{\partial x} = \frac{\partial (V_2 / V)}{\partial x} = \frac{1}{V} \frac{\partial V_2}{\partial x}$$

$$= \frac{1}{V} \frac{\partial}{\partial x} \frac{1}{6} (x y_3 z_4 - x z_3 y_4 - y x_3 z_4 + y z_3 x_4 + z x_3 y_4 - z y_3 x_4$$

$$- x_1 y_3 z_4 + x_1 z_3 y_4 + x_1 y z_4 - x_1 y z_3 - x_1 z y_4 + x_1 z y_3$$

$$+ y_1 x_3 z_4 - y_1 z_3 x_4 - y_1 x z_4 + y_1 x z_3 + y_1 z x_4 - y_1 z x_3$$

$$- z_1 x_3 y_4 + z_1 y_3 x_4 + z_1 x y_4 - z_1 x y_3 - z_1 y x_4 + z_1 y x_3)$$

$$= \frac{1}{6V} (y_3 z_4 - z_3 y_4 - y_1 z_4 + y_1 z_3 + z_1 y_4 - z_1 y_3)$$

$$\frac{\partial N_2}{\partial y}=\frac{\partial L_2}{\partial y}=\frac{\partial (V_2/V)}{\partial y}=\frac{1}{V}\frac{\partial V_2}{\partial y}$$

$$=\frac{1}{V}\frac{\partial}{\partial y}\frac{1}{6}(xy_3z_4-xz_3y_4-yx_3z_4+yz_3x_4+zx_3y_4-zy_3x_4$$

$$-x_1y_3z_4+x_1z_3y_4+x_1yz_4-x_1yz_3-x_1zy_4+x_1zy_3$$

$$+y_1x_3z_4-y_1z_3x_4-y_1xz_4+y_1xz_3+y_1zx_4-y_1zx_3$$

$$-z_1x_3y_4+z_1y_3x_4+z_1xy_4-z_1xy_3-z_1yx_4+z_1yx_3)$$

$$=\frac{1}{6V}(-x_3z_4+z_3x_4+x_1z_4-x_1z_3-z_1x_4+z_1x_3)$$

$$\frac{\partial N_2}{\partial z}=\frac{\partial L_2}{\partial z}=\frac{\partial (V_2/V)}{\partial z}=\frac{1}{V}\frac{\partial V_2}{\partial z}$$

$$=\frac{1}{V}\frac{\partial}{\partial z}\frac{1}{6}(xy_3z_4-xz_3y_4-yx_3z_4+yz_3x_4+zx_3y_4-zy_3x_4$$

$$-x_1y_3z_4+x_1z_3y_4+x_1yz_4-x_1yz_3-x_1zy_4+x_1zy_3$$

$$+y_1x_3z_4-y_1z_3x_4-y_1xz_4+y_1xz_3+y_1zx_4-y_1zx_3$$

$$-z_1x_3y_4+z_1y_3x_4+z_1xy_4-z_1xy_3-z_1yx_4+z_1yx_3)$$

$$=\frac{1}{6V}(x_3y_4-y_3x_4-x_1y_4+x_1y_3+y_1x_4-y_1x_3)$$

形状関数 N_3 の微分

$$\frac{\partial N_3}{\partial x}=\frac{\partial L_3}{\partial x}=\frac{\partial (V_3/V)}{\partial x}=\frac{1}{V}\frac{\partial V_3}{\partial x}$$

$$=\frac{1}{V}\frac{\partial}{\partial x}\frac{1}{6}(x_2yz_4-x_2zy_4-y_2xz_4+y_2zx_4+z_2xy_4-z_2yx_4$$

$$-x_1yz_4+x_1zy_4+x_1y_2z_4-x_1y_2z-x_1z_2y_4+x_1z_2y$$

$$+y_1xz_4-y_1zx_4-y_1x_2z_4+y_1x_2z+y_1z_2x_4-y_1z_2x$$

$$-z_1xy_4+z_1yx_4+z_1x_2y_4-z_1x_2y-z_1y_2x_4+z_1y_2x)$$

$$=\frac{1}{6V}(-y_2z_4+z_2y_4+y_1z_4-y_1z_2-z_1y_4+z_1y_2)$$

$$\frac{\partial N_3}{\partial y}=\frac{\partial L_3}{\partial y}=\frac{\partial(V_3/V)}{\partial y}=\frac{1}{V}\frac{\partial V_3}{\partial y}$$

$$=\frac{1}{V}\frac{\partial}{\partial y}\frac{1}{6}(x_2yz_4-x_2zy_4-y_2xz_4+y_2zx_4+z_2xy_4-z_2yx_4$$

$$-x_1yz_4+x_1zy_4+x_1y_2z_4-x_1y_2z-x_1z_2y_4+x_1z_2y$$

$$+y_1xz_4-y_1zx_4-y_1x_2z_4+y_1x_2z+y_1z_2x_4-y_1z_2x$$

$$-z_1xy_4+z_1yx_4+z_1x_2y_4-z_1x_2y-z_1y_2x_4+z_1y_2x)$$

$$=\frac{1}{6V}(x_2z_4-z_2x_4-x_1z_4+x_1z_2+z_1x_4-z_1x_2)$$

$$\frac{\partial N_3}{\partial z}=\frac{\partial L_3}{\partial z}=\frac{\partial(V_3/V)}{\partial z}=\frac{1}{V}\frac{\partial V_3}{\partial z}$$

$$=\frac{1}{V}\frac{\partial}{\partial z}\frac{1}{6}(x_2yz_4-x_2zy_4-y_2xz_4+y_2zx_4+z_2xy_4-z_2yx_4$$

$$-x_1yz_4+x_1zy_4+x_1y_2z_4-x_1y_2z-x_1z_2y_4+x_1z_2y$$

$$+y_1xz_4-y_1zx_4-y_1x_2z_4+y_1x_2z+y_1z_2x_4-y_1z_2x$$

$$-z_1xy_4+z_1yx_4+z_1x_2y_4-z_1x_2y-z_1y_2x_4+z_1y_2x)$$

$$=\frac{1}{6V}(-x_2y_4+y_2x_4+x_1y_4-x_1y_2-y_1x_4+y_1x_2)$$

形状関数 N_4 の微分

$$\frac{\partial N_4}{\partial x}=\frac{\partial L_4}{\partial x}=\frac{\partial(V_4/V)}{\partial x}=\frac{1}{V}\frac{\partial V_4}{\partial x}$$

$$=\frac{1}{V}\frac{\partial}{\partial x}\frac{1}{6}(x_2y_3z-x_2z_3y-y_2x_3z+y_2z_3x+z_2x_3y-z_2y_3x$$

$$-x_1y_3z+x_1z_3y+x_1y_2z-x_1y_2z_3-x_1z_2y+x_1z_2y_3$$

$$+y_1x_3z-y_1z_3x-y_1x_2z+y_1x_2z_3+y_1z_2x-y_1z_2x_3$$

$$-z_1x_3y+z_1y_3x+z_1x_2y-z_1x_2y_3-z_1y_2x+z_1y_2x_3)$$

$$=\frac{1}{6V}(y_2z_3-z_2y_3-y_1z_3+y_1z_2+z_1y_3-z_1y_2)$$

$$\frac{\partial N_4}{\partial y} = \frac{\partial L_4}{\partial y} = \frac{\partial (V_4 / V)}{\partial y} = \frac{1}{V}\frac{\partial V_4}{\partial y}$$

$$= \frac{1}{V}\frac{\partial}{\partial y}\frac{1}{6}(x_2 y_3 z - x_2 z_3 y - y_2 x_3 z + y_2 z_3 x + z_2 x_3 y - z_2 y_3 x$$

$$- x_1 y_3 z + x_1 z_3 y + x_1 y_2 z - x_1 y_2 z_3 - x_1 z_2 y + x_1 z_2 y_3$$

$$+ y_1 x_3 z - y_1 z_3 x - y_1 x_2 z + y_1 x_2 z_3 + y_1 z_2 x - y_1 z_2 x_3$$

$$- z_1 x_3 y + z_1 y_3 x + z_1 x_2 y - z_1 x_2 y_3 - z_1 y_2 x + z_1 y_2 x_3)$$

$$= \frac{1}{6V}(-x_2 z_3 + z_2 x_3 + x_1 z_3 - x_1 z_2 - z_1 x_3 + z_1 x_2)$$

$$\frac{\partial N_4}{\partial z} = \frac{\partial L_4}{\partial z} = \frac{\partial (V_4 / V)}{\partial z} = \frac{1}{V}\frac{\partial V_4}{\partial z}$$

$$= \frac{1}{V}\frac{\partial}{\partial z}\frac{1}{6}(x_2 y_3 z - x_2 z_3 y - y_2 x_3 z + y_2 z_3 x + z_2 x_3 y - z_2 y_3 x$$

$$- x_1 y_3 z + x_1 z_3 y + x_1 y_2 z - x_1 y_2 z_3 - x_1 z_2 y + x_1 z_2 y_3$$

$$+ y_1 x_3 z - y_1 z_3 x - y_1 x_2 z + y_1 x_2 z_3 + y_1 z_2 x - y_1 z_2 x_3$$

$$- z_1 x_3 y + z_1 y_3 x + z_1 x_2 y - z_1 x_2 y_3 - z_1 y_2 x + z_1 y_2 x_3)$$

$$= \frac{1}{6V}(x_2 y_3 - y_2 x_3 - x_1 y_3 + x_1 y_2 + y_1 x_3 - y_1 x_2)$$

形状関数の微分まとめ

$$\frac{\partial N_1}{\partial x} = \frac{\partial L_1}{\partial x} = \frac{c_{1x}}{6V} = \frac{1}{6V}(-y_3 z_4 + z_3 y_4 + y_2 z_4 - y_2 z_3 - z_2 y_4 + z_2 y_3)$$

$$\frac{\partial N_1}{\partial y} = \frac{\partial L_1}{\partial y} = \frac{c_{1y}}{6V} = \frac{1}{6V}(x_3 z_4 - z_3 x_4 - x_2 z_4 + x_2 z_3 + z_2 x_4 - z_2 x_3)$$

$$\frac{\partial N_1}{\partial z} = \frac{\partial L_1}{\partial z} = \frac{c_{1z}}{6V} = \frac{1}{6V}(-x_3 y_4 + y_3 x_4 + x_2 y_4 - x_2 y_3 - y_2 x_4 + y_2 x_3)$$

$$\frac{\partial N_2}{\partial x} = \frac{\partial L_2}{\partial x} = \frac{c_{2x}}{6V} = \frac{1}{6V}(y_3 z_4 - z_3 y_4 - y_1 z_4 + y_1 z_3 + z_1 y_4 - z_1 y_3)$$

$$\frac{\partial N_2}{\partial y} = \frac{\partial L_2}{\partial y} = \frac{c_{2y}}{6V} = \frac{1}{6V}(-x_3 z_4 + z_3 x_4 + x_1 z_4 - x_1 z_3 - z_1 x_4 + z_1 x_3)$$

$$\frac{\partial N_2}{\partial z} = \frac{\partial L_2}{\partial z} = \frac{c_{2z}}{6V} = \frac{1}{6V}(x_3 y_4 - y_3 x_4 - x_1 y_4 + x_1 y_3 + y_1 x_4 - y_1 x_3)$$

$$\frac{\partial N_3}{\partial x} = \frac{\partial L_3}{\partial x} = \frac{c_{3x}}{6V} = \frac{1}{6V}(-y_2 z_4 + z_2 y_4 + y_1 z_4 - y_1 z_2 - z_1 y_4 + z_1 y_2)$$

$$\frac{\partial N_3}{\partial y} = \frac{\partial L_3}{\partial y} = \frac{c_{3y}}{6V} = \frac{1}{6V}(x_2 z_4 - z_2 x_4 - x_1 z_4 + x_1 z_2 + z_1 x_4 - z_1 x_2)$$

$$\frac{\partial N_3}{\partial z} = \frac{\partial L_3}{\partial z} = \frac{c_{3z}}{6V} = \frac{1}{6V}(-x_2 y_4 + y_2 x_4 + x_1 y_4 - x_1 y_2 - y_1 x_4 + y_1 x_2)$$

$$\frac{\partial N_4}{\partial x} = \frac{\partial L_4}{\partial x} = \frac{c_{4x}}{6V} = \frac{1}{6V}(y_2 z_3 - z_2 y_3 - y_1 z_3 + y_1 z_2 + z_1 y_3 - z_1 y_2)$$

$$\frac{\partial N_4}{\partial y} = \frac{\partial L_4}{\partial y} = \frac{c_{4y}}{6V} = \frac{1}{6V}(-x_2 z_3 + z_2 x_3 + x_1 z_3 - x_1 z_2 - z_1 x_3 + z_1 x_2)$$

$$\frac{\partial N_4}{\partial z} = \frac{\partial L_4}{\partial z} = \frac{c_{4z}}{6V} = \frac{1}{6V}(x_2 y_3 - y_2 x_3 - x_1 y_3 + x_1 y_2 + y_1 x_3 - y_1 x_2)$$

・離散化

数値計算の目的は、解析的に解けない収支式を四則演算に分解し、プログラム化して解くことです。有限要素法もその１つです。四則演算に分解することを離散化と言います。

離散化

$$\phi = \frac{\partial P}{\partial \tau} + V_x \frac{\partial P}{\partial x} + V_y \frac{\partial P}{\partial y} + \frac{1}{Ma^2}(\frac{\partial V_x}{\partial X} + \frac{\partial V_y}{\partial Y}) = 0$$

$$\int_V [N]^T \phi dV = \int_V \begin{bmatrix} N_1 \\ N_1 \\ N_3 \end{bmatrix} \phi dV$$

$$= \int_V \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} dV$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x1} + N_2 V_{x2} + N_3 V_{x3}$$

$$= \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \{V_{x1} \quad V_{x2} \quad V_{x3}\}$$

$$= [N]^T \{V_x\}$$

$$V_y = [N]^T \{V_y\}$$

$$P = [N]^T \{P\}$$

$$\int_S [N]^T \phi dS$$

$$= \int_S [N]^T \{\frac{\partial P}{\partial \tau} + V_x \frac{\partial P}{\partial x} + V_y \frac{\partial P}{\partial y} + \frac{1}{Ma^2}(\frac{\partial V_x}{\partial X} + \frac{\partial V_y}{\partial Y})\} dS$$

$$= \int_S [N]^T \frac{\partial P}{\partial \tau} dS + \int_S [N]^T V_x \frac{\partial P}{\partial X} dS + \int_S [N]^T V_y \frac{\partial P}{\partial Y} dS$$

$$+ \frac{1}{Ma^2} \int_S [N]^T \frac{\partial V_x}{\partial X} dS + \frac{1}{Ma^2} \int_S [N]^T \frac{\partial V_y}{\partial Y} dS$$

$$
\begin{aligned}
&= \int_S [N]^T [N] dS \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^T \frac{\partial [N]\{P\}^{\Delta\tau+\tau}}{\partial X} dS + V_y \int_S [N]^T \frac{\partial [N]\{P\}^{\Delta\tau+\tau}}{\partial Y} dS \\
&+ \frac{1}{Ma^2} \int_S [N]^T \frac{\partial [N]\{V_x\}^{\Delta\tau+\tau}}{\partial X} dS + \frac{1}{Ma^2} \int_S [N]^T \frac{\partial [N]\{V_y\}^{\Delta\tau+\tau}}{\partial Y} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^T [N] dS \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^T \frac{\partial [N]}{\partial X} dS \{P\}^{\Delta\tau+\tau} + V_y \int_S [N]^T \frac{\partial [N]}{\partial Y} dS \{P\}^{\Delta\tau+\tau} \\
&+ \frac{1}{Ma^2} \int_S [N]^T \frac{\partial [N]}{\partial X} dS \{V_x\}^{\Delta\tau+\tau} + \frac{1}{Ma^2} \int_S [N]^T \frac{\partial [N]}{\partial Y} dS \{V_y\}^{\Delta\tau+\tau}
\end{aligned}
$$

$$=\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} dS \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}$$

$$+V_x \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X} \end{bmatrix} dS \{P\}^{\Delta\tau+\tau}$$

$$+V_y \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y} \end{bmatrix} dS \{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2} \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X} \end{bmatrix} dS \{V_x\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2} \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y} \end{bmatrix} dS \{V_y\}^{\Delta\tau+\tau}$$

$$
=\int_S \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2N_2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3N_3 \end{bmatrix} dS \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}
$$

$$
+V_x \int_S \begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} \end{bmatrix} dS \{V_x\}^{\Delta\tau+\tau}
$$

$$
+V_y \int_S \begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} \end{bmatrix} dS \{V_y\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2} \int_S \begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} \end{bmatrix} dS \{V_x\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2} \int_S \begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} \end{bmatrix} dS \{V_y\}^{\Delta\tau+\tau}
$$

$$=\int_S \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2L_2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3L_3 \end{bmatrix} dS \frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}$$

$$+V_x\int_S \begin{bmatrix} L_1\frac{\partial L_1}{\partial X} & L_1\frac{\partial L_2}{\partial X} & L_1\frac{\partial L_3}{\partial X} \\ L_2\frac{\partial L_1}{\partial X} & L_2\frac{\partial L_2}{\partial X} & L_2\frac{\partial L_3}{\partial X} \\ L_3\frac{\partial L_1}{\partial X} & L_3\frac{\partial L_2}{\partial X} & L_3\frac{\partial L_3}{\partial X} \end{bmatrix} dS\{P\}^{\Delta\tau+\tau}$$

$$+V_y\int_S \begin{bmatrix} L_1\frac{\partial L_1}{\partial Y} & L_1\frac{\partial L_2}{\partial Y} & L_1\frac{\partial L_3}{\partial Y} \\ L_2\frac{\partial L_1}{\partial Y} & L_2\frac{\partial L_2}{\partial Y} & L_2\frac{\partial L_3}{\partial Y} \\ L_3\frac{\partial L_1}{\partial Y} & L_3\frac{\partial L_2}{\partial Y} & L_3\frac{\partial L_3}{\partial Y} \end{bmatrix} dS\{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\int_S \begin{bmatrix} L_1\frac{\partial L_1}{\partial X} & L_1\frac{\partial L_2}{\partial X} & L_1\frac{\partial L_3}{\partial X} \\ L_2\frac{\partial L_1}{\partial X} & L_2\frac{\partial L_2}{\partial X} & L_2\frac{\partial L_3}{\partial X} \\ L_3\frac{\partial L_1}{\partial X} & L_3\frac{\partial L_2}{\partial X} & L_3\frac{\partial L_3}{\partial X} \end{bmatrix} dS\{V_x\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\int_S \begin{bmatrix} L_1\frac{\partial L_1}{\partial Y} & L_1\frac{\partial L_2}{\partial Y} & L_1\frac{\partial L_3}{\partial Y} \\ L_2\frac{\partial L_1}{\partial Y} & L_2\frac{\partial L_2}{\partial Y} & L_2\frac{\partial L_3}{\partial Y} \\ L_3\frac{\partial L_1}{\partial Y} & L_3\frac{\partial L_2}{\partial Y} & L_3\frac{\partial L_3}{\partial Y} \end{bmatrix} dS\{V_y\}^{\Delta\tau+\tau}$$

$$= \int_S \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2L_2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3L_3 \end{bmatrix} dS \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_S \frac{1}{2A} \begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} \end{bmatrix} dS \{P\}^{\Delta\tau+\tau}$$

$$+ V_y \int_S \frac{1}{2A} \begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} \end{bmatrix} dS \{P\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_S \frac{1}{2A} \begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} \end{bmatrix} dS \{V_x\}^{\Delta\tau+\tau} 3$$

$$+ \frac{1}{Ma^2} \int_S \frac{1}{2A} \begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} \end{bmatrix} dS \{V_y\}^{\Delta\tau+\tau}$$

ここで、

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!} 2A$$

$$\int_S L_i L_j dS = \begin{cases} \dfrac{1!1!}{(1+1+2)!} 2A = \dfrac{2}{4!} A = \dfrac{1}{12} A & (i \neq j) \\ \dfrac{2!}{(1+1+2)!} 2A = \dfrac{4}{4!} V = \dfrac{1}{6} A & (i = j) \end{cases}$$

$$\int_S L_i dS = \frac{1!}{(1+2)!} 2A = \frac{2}{3!} A = \frac{1}{3} A$$

なので、

$$=\frac{A}{12}\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}$$

$$+V_x\frac{1}{2A}\frac{A}{3}\begin{bmatrix}c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\end{bmatrix}\{P\}^{\Delta\tau+\tau}$$

$$+V_y\frac{1}{2A}\frac{A}{3}\begin{bmatrix}c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\end{bmatrix}\{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\frac{1}{2A}\frac{A}{3}\begin{bmatrix}c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\end{bmatrix}\{V_x\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\frac{1}{2A}\frac{A}{3}\begin{bmatrix}c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\end{bmatrix}\{V_y\}^{\Delta\tau+\tau}$$

$$=\frac{A}{12}\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}$$

$$+V_x\frac{1}{6}\begin{bmatrix}c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\end{bmatrix}\{P\}^{\Delta\tau+\tau}$$

$$+V_y\frac{1}{6}\begin{bmatrix}c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\end{bmatrix}\{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\frac{1}{6}\begin{bmatrix}c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\\ c_{1x} & c_{2x} & c_{3x}\end{bmatrix}\{V_x\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\frac{1}{6}\begin{bmatrix}c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\\ c_{1y} & c_{2y} & c_{3y}\end{bmatrix}\{V_y\}^{\Delta\tau+\tau}$$

$$=[C]\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}$$

$$+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}$$

$$=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$[C]\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}=[0]$$

$$
\begin{aligned}
&[C]\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}\\
&+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}\\
&+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}=[0]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{[C]}{\Delta\tau}\{P\}^{\Delta\tau+\tau}\\
&+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}\\
&+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}=\frac{[C]}{\Delta\tau}\{P\}^{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&(\frac{[C]}{\Delta\tau}+V_x[C_x]+V_y[C_y])\{P\}^{\Delta\tau+\tau}\\
&+\frac{1}{Ma^2}([C_x]\{V_x\}^{\Delta\tau+\tau}+[C_y]\{V_y\}^{\Delta\tau+\tau})=\frac{[C]}{\Delta\tau}\{P\}^{\tau}
\end{aligned}
$$

・運動量収支式

$$\phi_x = \frac{\partial V_x}{\partial \tau} + V_x \frac{\partial V_x}{\partial X} + V_y \frac{\partial V_x}{\partial Y} - \frac{\partial \sigma_{xx}^*}{\partial X} - \frac{\partial \sigma_{yx}^*}{\partial Y} - g_x^* = 0$$

$$\phi_y = \frac{\partial V_y}{\partial \tau} + V_x \frac{\partial V_y}{\partial X} + V_y \frac{\partial V_y}{\partial Y} - \frac{\partial \sigma_{xy}^*}{\partial X} - \frac{\partial \sigma_{yy}^*}{\partial Y} - g_y^* = 0$$

要素に 3 角形を用いた場合、

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \phi_i dV = \int_V \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} dV = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x,1} + N_2 V_{x,2} + N_3 V_{x,3} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} V_{x,1} \\ V_{x,2} \\ V_{x,3} \end{Bmatrix} = \begin{bmatrix} N \end{bmatrix} \begin{Bmatrix} V_x \end{Bmatrix} = 0$$

$$V_y = N_1 V_{y,1} + N_2 V_{y,2} + N_3 V_{y,3} = 0$$

$$P = N_1 P_{z,1} + N_2 P_{z,2} + N_3 P_{z,3}$$

となるので、

$$\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \left(\frac{\partial V_i}{\partial \tau} + V_x \frac{\partial V_i}{\partial X} + V_y \frac{\partial V_i}{\partial Y} - \frac{\partial \sigma_{xi}^*}{\partial X} - \frac{\partial \sigma_{yi}^*}{\partial Y} - g_i^*\right) dS$$

$$= \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial \tau} dS + V_x \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial X} dS + V_y \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial Y} dS$$

$$- \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \sigma_{xi}^*}{\partial X} dS - \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \sigma_{yi}^*}{\partial Y} dS - \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} g_i^* dS$$

展開すると

$$\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \left(\frac{\partial V_i}{\partial \tau} + V_x \frac{\partial V_i}{\partial X} + V_y \frac{\partial V_i}{\partial Y} + \frac{\partial \sigma_{xi}^*}{\partial X} + \frac{\partial \sigma_{yi}^*}{\partial Y} - g_i^*\right) dS$$

$$= \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial \tau} dS + V_x \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial X} dS + V_y \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial V_i}{\partial Y} dS$$

$$- \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \sigma_{xi}^*}{\partial X} dS - \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \sigma_{yi}^*}{\partial Y} dS$$

$$- \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} g_i^* dS$$

$$= \int_S [N]^T \frac{\partial V_i}{\partial \tau} dS + V_x \int_S [N]^T \frac{\partial V_i}{\partial X} dS + V_y \int_S [N]^T \frac{\partial V_i}{\partial Y} dS$$

$$- \int_S [N]^T \frac{\partial \sigma_{xi}^*}{\partial X} dS - \int_S [N]^T \frac{\partial \sigma_{yi}^*}{\partial Y} dS$$

$$- \int_S [N]^T g_i^* dS$$

グリーン・ガウスの定理より

$$= \int_S [N]^{\mathrm{T}} \frac{[N](\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau})}{\Delta \tau} dS$$

$$+ V_x \int_S [N]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial X} dS + V_y \int_S [N]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial Y} dS$$

$$- \int_L [N]^{\mathrm{T}} \sigma_{xi}^* n_x dL + \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \sigma_{xi}^* dS$$

$$- \int_L [N]^{\mathrm{T}} \sigma_{yi}^* n_y dL + \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \sigma_{yi}^* dS$$

$$- \int_S [N]^{\mathrm{T}} g_i^* dS$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}} [N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \sigma_{xi}^{*} dS + \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \sigma_{yi}^{*} dS \\
&- \int_L [N]^{\mathrm{T}} (\sigma_{xi}^{*} n_x + \sigma_{yi}^{*} n_y) dL \\
&- \int_S [N]^{\mathrm{T}} g_i^{*} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}} [N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \{-\delta_{xi} P + \frac{1}{Re} (\frac{\partial V_i}{\partial X} + \frac{\partial V_x}{\partial X_i})\} dS \\
&+ \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \{-\delta_{yi} P + \frac{1}{Re} (\frac{\partial V_i}{\partial Y} + \frac{\partial V_y}{\partial X_i})\} dS \\
&- \frac{-2K^{*}}{We} n_i \int_L [N]^{\mathrm{T}} dL \\
&- \int_S [N]^{\mathrm{T}} g_i^{*} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}}[N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \{-\delta_{xi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial X} + \frac{\partial V_x}{\partial X_i})\} dS \\
&+ \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \{-\delta_{yi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial Y} + \frac{\partial V_y}{\partial X_i})\} dS \\
&+ \frac{2K^*}{We} n_i \int_L [N]^{\mathrm{T}} dL \\
&- \int_S [N]^{\mathrm{T}} g_i^* dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}}[N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial V_i}{\partial X} dS + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial V_x}{\partial X_i} dS - \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \delta_{xi} P dS \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial V_i}{\partial Y} dS + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial V_y}{\partial X_i} dS - \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \delta_{yi} P dS \\
&+ \frac{2K^*}{We} n_i \int_L [N]^{\mathrm{T}} dL \\
&- g_i^* \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}}[N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial X} dS + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{V_x\}}{\partial X_i} dS - \delta_{xi} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} [N]\{P\} dS \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial Y} dS + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{V_y\}}{\partial X_i} dS - \delta_{yi} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} [N]\{P\} dS \\
&+ \frac{2K^*}{We} n_i L \int_L [N]^{\mathrm{T}} dL \\
&- g_i^* \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}}[N] dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{V_i\} + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X_i}\right] dS \{V_x\} - \delta_{xi} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} [N] dS \{P\} \\
&+ \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{V_i\} + \frac{1}{Re} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X_i}\right] dS \{V_y\} - \delta_{yi} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} [N] dS \{P\} \\
&+ \frac{2K^*}{We} n_i \int_L [N]^{\mathrm{T}} dL \\
&- g_i^* \int_S [N]^{\mathrm{T}} dS \qquad (i = 1,2)
\end{aligned}
$$

蓄積量

$$
= \int_S \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2N_2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3N_3 \end{bmatrix} dS \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}
$$

対流項

$$+V_x\int_S\begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} \end{bmatrix}dS\{V_i\}$$

$$+V_y\int_S\begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} \end{bmatrix}dS\{V_i\}$$

粘性項の x 成分

$$+\frac{1}{Re}\int_S\begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X} \end{bmatrix}dS\{V_i\}$$

$$+\frac{1}{Re}\int_S\begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X_i} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X_i} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X_i} \end{bmatrix}dS\{V_x\}$$

$$-\delta_{xi}\int_S\begin{bmatrix} \frac{\partial N_1}{\partial X}N_1 & \frac{\partial N_1}{\partial X}N_2 & \frac{\partial N_1}{\partial X}N_3 \\ \frac{\partial N_2}{\partial X}N_1 & \frac{\partial N_2}{\partial X}N_2 & \frac{\partial N_2}{\partial X}N_3 \\ \frac{\partial N_3}{\partial X}N_1 & \frac{\partial N_3}{\partial X}N_2 & \frac{\partial N_3}{\partial X}N_3 \end{bmatrix}dS\{P\}$$

圧力項の x 成分

$$+\frac{1}{Re}\int_S \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial Y} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial Y} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial Y} \end{bmatrix} dS\{V_i\}$$

粘性項の y 成分

$$+\frac{1}{Re}\int_S \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial X_i} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial X_i} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial X_i} \end{bmatrix} dS\{V_y\}$$

$$-\int_S \begin{bmatrix} \frac{\partial N_1}{\partial Y}N_1 & \frac{\partial N_1}{\partial Y}N_2 & \frac{\partial N_1}{\partial Y}N_3 \\ \frac{\partial N_2}{\partial Y}N_1 & \frac{\partial N_2}{\partial Y}N_2 & \frac{\partial N_2}{\partial Y}N_3 \\ \frac{\partial N_3}{\partial Y}N_1 & \frac{\partial N_3}{\partial Y}N_2 & \frac{\partial N_3}{\partial Y}N_3 \end{bmatrix} dS\{P\}$$

圧力項の y 成分

$$+\frac{2K^*}{We}n_i\int_L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dL$$

表面張力項、重力項

$$-g_i^*\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dS \qquad (i=1,2)$$

$$=\int_S \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2L_2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3L_3 \end{bmatrix} dS \frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}$$

$$+V_x\int_S \frac{1}{2A}\begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} \end{bmatrix} dS\{V_i\}$$

$$+V_y\int_S \frac{1}{2A}\begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} \end{bmatrix} dS\{V_i\}$$

$$+\frac{1}{Re}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix} dS\{V_i\}$$

$$+\frac{1}{Re}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} \end{bmatrix} dS\{V_x\}$$

$$-\delta_{xi}\int_S \frac{1}{2A}\begin{bmatrix} c_{1x}L_1 & c_{1x}L_2 & c_{1x}L_3 \\ c_{2x}L_1 & c_{2x}L_2 & c_{2x}L_3 \\ c_{3x}L_1 & c_{3x}L_2 & c_{3x}L_3 \end{bmatrix} dS\{P\}$$

$$+\frac{1}{Re}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix} dS\{V_i\}$$

$$+\frac{1}{Re}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} \end{bmatrix} dS\{V_y\}$$

$$-\delta_{yi}\int_S \frac{1}{2A}\begin{bmatrix} c_{1y}L_1 & c_{1y}L_2 & c_{1y}L_3 \\ c_{2y}L_1 & c_{2y}L_2 & c_{2y}L_3 \\ c_{3y}L_1 & c_{3y}L_2 & c_{3y}L_3 \end{bmatrix} dS\{P\}$$

$$+\frac{2K^*}{We}n_i\int_L \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} dL$$

$$-g_i^* \int_S \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}^{\mathrm{T}} dS \qquad\qquad (i=1,2)$$

ここで、

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!} 2A$$

$$\int_S L_i L_j dS = \begin{cases} \dfrac{1!1!}{(1+1+2)!} 2A = \dfrac{2}{4!} A = \dfrac{1}{12} A & (i \neq j) \\ \dfrac{2!}{(1+1+2)!} 2A = \dfrac{4}{4!} V = \dfrac{1}{6} A & (i = j) \end{cases}$$

$$\int_S L_i dS = \frac{1!}{(1+2)!} 2A = \frac{2}{3!} A = \frac{1}{3} A$$

$$\int_L L_1^p L_2^q dL = \frac{p!q!}{(p+q)!} L$$

$$\int_L L_i dL = \frac{1}{1!} L = L$$

$$= \frac{1}{12} A \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+V_x \frac{1}{2A} \frac{A}{3} \begin{bmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \end{bmatrix} \{V_i\}$$

$$+V_y \frac{1}{2A} \frac{A}{3} \begin{bmatrix} c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \end{bmatrix} \{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A^2}A\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A^2}A\begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} \end{bmatrix}\{V_x\}$$

$$-\delta_{xi}\frac{1}{2A}\frac{A}{3}\begin{bmatrix} c_{1x} & c_{1x} & c_{1x} \\ c_{2x} & c_{2x} & c_{2x} \\ c_{3x} & c_{3x} & c_{3x} \end{bmatrix}\{P\}$$

$$+\frac{1}{Re}\frac{1}{4A^2}A\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A^2}A\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} \end{bmatrix}\{V_y\}$$

$$-\delta_{yi}\frac{1}{2A}\frac{A}{3}\begin{bmatrix} c_{1y} & c_{1y} & c_{1y} \\ c_{2y} & c_{2y} & c_{2y} \\ c_{3y} & c_{3y} & c_{3y} \end{bmatrix}\{P\}$$

$$+\frac{2K^*}{We}n_iL\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-g_i^*\frac{A}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (i=1,2)$$

$$=\frac{1}{12}A\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}$$

$$+V_x\frac{1}{6}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \end{bmatrix}\{V_i\}$$

$$+V_y\frac{1}{6}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A}\begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} \end{bmatrix}\{V_x\}$$

$$-\delta_{xi}\frac{1}{6}\begin{bmatrix} c_{1x} & c_{1x} & c_{1x} \\ c_{2x} & c_{2x} & c_{2x} \\ c_{3x} & c_{3x} & c_{3x} \end{bmatrix}\{P\}$$

$$+\frac{1}{Re}\frac{1}{4A}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{4A}\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} \end{bmatrix}\{V_y\}$$

$$-\delta_{yi}\frac{1}{6}\begin{bmatrix} c_{1y} & c_{1y} & c_{1y} \\ c_{2y} & c_{2y} & c_{2y} \\ c_{3y} & c_{3y} & c_{3y} \end{bmatrix}\{P\}$$

$$+\frac{2K^*}{We}n_iL\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-g_i^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad (i=1,2)$$

$$=[C]\frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}+V_x[C_x]\{V_i\}+V_y[C_y]\{V_i\}$$
$$+\frac{1}{Re}[S_{xx}]\{V_i\}+\frac{1}{Re}[S_{xi}]\{V_x\}-\delta_{xi}[H_x]\{P\}$$
$$+\frac{1}{Re}[S_{yy}]\{V_i\}+\frac{1}{Re}[S_{yi}]\{V_y\}-\delta_{yi}[H_y]\{P\}$$
$$+\frac{2K^*}{We}Ln_i\begin{bmatrix}1\\1\end{bmatrix}-g_i^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad (i=1,2)$$

$$=[C]\frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}+(V_x[C_x]+V_y[C_y])\{V_i\}$$
$$+\frac{1}{Re}([S_{xx}]+[S_{yy}])\{V_i\}$$
$$+\frac{1}{Re}([S_{xi}]\{V_x\}+[S_{yi}]\{V_y\})$$
$$-(\delta_{xi}[H_x]+\delta_{yi}[H_y])\{P\}$$
$$+\frac{2K^*}{We}Ln_i\begin{bmatrix}1\\1\end{bmatrix}$$
$$-g_i^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad (i=1,2,3)$$
$$=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$[C]\frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}+(V_x[C_x]+V_y[C_y])\{V_i\}$$
$$+\frac{1}{Re}([S_{xx}]+[S_{yy}])\{V_i\}$$
$$+\frac{1}{Re}([S_{xi}]\{V_x\}+[S_{yi}]\{V_y\})$$
$$-(\delta_{xi}[H_x]+\delta_{yi}[H_y])\{P\}$$
$$+\frac{2K^*}{We}Ln_i\begin{bmatrix}1\\1\end{bmatrix}-g_i^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\qquad (i=1,2)$$

陰解法

$$\frac{[C]}{\Delta\tau}\{V_i\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y])\{V_i\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}([S_{xx}]+[S_{yy}])\{V_i\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}([S_{xi}]\{V_x\}^{\tau+\Delta\tau}+[S_{yi}]\{V_y\}^{\tau+\Delta\tau})$$
$$-(\delta_{xi}[H_x]+\delta_{yi}[H_y])\{P\}^{\tau+\Delta\tau}$$
$$=\frac{[C]}{\Delta\tau}\{V_i\}^{\tau}-\frac{2K^*}{We}Ln_i\begin{bmatrix}1\\1\end{bmatrix}+g_i^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad (i=1,2)$$

x 成分

$$\frac{[C]}{\Delta\tau}\{V_x\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y])\{V_x\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}\{(2[S_{xx}]+[S_{yy}])\{V_x\}^{\tau+\Delta\tau}+[S_{yx}]\{V_y\}^{\tau+\Delta\tau}-[H_x]\{P\}^{\tau+\Delta\tau}\}$$
$$=\frac{[C]}{\Delta\tau}\{V_x\}^{\tau}-\frac{2K^*}{We}Ln_x\begin{bmatrix}1\\1\end{bmatrix}+g_x^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

y成分

$$\frac{[C]}{\Delta\tau}\{V_y\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y])\{V_y\}^{\tau+\Delta\tau}$$

$$+\frac{1}{Re}\{[S_{xy}]\{V_x\}^{\tau+\Delta\tau}+([S_{xx}]+2[S_{yy}])\{V_y\}^{\tau+\Delta\tau}-[H_y]\{P\}^{\tau+\Delta\tau}\}$$

$$=\frac{[C]}{\Delta\tau}\{V_y\}^{\tau}-\frac{2K^*}{We}Ln_y\begin{bmatrix}1\\1\end{bmatrix}+g_y^*\frac{A}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

離散化

$$\phi = \frac{\partial P}{\partial \tau} + V_x \frac{\partial P}{\partial x} + V_y \frac{\partial P}{\partial y} + V_z \frac{\partial P}{\partial z} + \frac{1}{Ma^2}(\frac{\partial V_x}{\partial X} + \frac{\partial V_y}{\partial Y} + \frac{\partial V_z}{\partial Z}) = 0$$

$$\int_V [N]^T \phi dV = \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \phi dV$$

$$= \int_V \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dV$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x1} + N_2 V_{x2} + N_3 V_{x3} + N_4 V_{x4}$$

$$= \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \{V_{x1} \quad V_{x2} \quad V_{x3} \quad V_{x4}\}$$

$$= [N]^T \{V_x\}$$

$$V_y = [N]^T \{V_y\}$$

$$V_z = [N]^T \{V_z\}$$

$$P = [N]^T \{P\}$$

$$\int_V [N]^T \phi dV$$

$$= \int_V [N]^T \{\frac{\partial P}{\partial \tau} + V_x \frac{\partial P}{\partial x} + V_y \frac{\partial P}{\partial y} + V_z \frac{\partial P}{\partial z} + \frac{1}{Ma^2}(\frac{\partial V_x}{\partial X} + \frac{\partial V_y}{\partial Y} + \frac{\partial V_z}{\partial Z})\} dV$$

$$= \int_V [N]^T \frac{\partial P}{\partial \tau} dV + \int_V [N]^T V_x \frac{\partial P}{\partial X} dV + \int_V [N]^T V_y \frac{\partial P}{\partial Y} dV + \int_V [N]^T V_z \frac{\partial P}{\partial Z} dV$$

$$+ \frac{1}{Ma^2} \int_V [N]^T \frac{\partial V_x}{\partial X} dV + \frac{1}{Ma^2} \int_V [N]^T \frac{\partial V_y}{\partial Y} dV + \frac{1}{Ma^2} \int_V [N]^T \frac{\partial V_z}{\partial Z} dV$$

$$= \int_V [N]^T [N] dV \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta \tau}$$

$$+ V_x \int_V [N]^T \frac{\partial [N]\{P\}^{\Delta\tau+\tau}}{\partial X} dV + V_y \int_V [N]^T \frac{\partial [N]\{P\}^{\Delta\tau+\tau}}{\partial Y} dV$$

$$+ V_z \int_V [N]^T \frac{\partial [N]\{P\}^{\Delta\tau+\tau}}{\partial Z} dV$$

$$+ \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]\{V_x\}^{\Delta\tau+\tau}}{\partial X} dV + \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]\{V_y\}^{\Delta\tau+\tau}}{\partial Y} dV$$

$$+ \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]\{V_z\}^{\Delta\tau+\tau}}{\partial Z} dV$$

$$= \int_V [N]^T [N] dV \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta \tau}$$

$$+ V_x \int_V [N]^T \frac{\partial [N]}{\partial X} dV \{P\}^{\Delta\tau+\tau} + V_y \int_V [N]^T \frac{\partial [N]}{\partial Y} dV \{P\}^{\Delta\tau+\tau}$$

$$+ V_z \int_V [N]^T \frac{\partial [N]}{\partial Z} dV \{P\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]}{\partial X} dV \{V_x\}^{\Delta\tau+\tau} + \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]}{\partial Y} dV \{V_y\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_V [N]^T \frac{\partial [N]}{\partial Z} dV \{V_z\}^{\Delta\tau+\tau}$$

$$= \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} dV \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X} & \frac{\partial N_4}{\partial X} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}$$

$$+ V_y \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y} & \frac{\partial N_4}{\partial Y} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}$$

$$+ V_z \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Z} & \frac{\partial N_2}{\partial Z} & \frac{\partial N_3}{\partial Z} & \frac{\partial N_4}{\partial Z} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X} & \frac{\partial N_4}{\partial X} \end{bmatrix} dV \{V_x\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y} & \frac{\partial N_4}{\partial Y} \end{bmatrix} dV \{V_y\}^{\Delta\tau+\tau}$$

$$+ \frac{1}{Ma^2} \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial Z} & \frac{\partial N_2}{\partial Z} & \frac{\partial N_3}{\partial Z} & \frac{\partial N_4}{\partial Z} \end{bmatrix} dV \{V_z\}^{\Delta\tau+\tau}$$

$$
= \int_V \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_2N_1 & N_2N_2 & N_2N_3 & N_2N_4 \\ N_3N_1 & N_3N_2 & N_3N_3 & N_3N_4 \\ N_4N_1 & N_4N_2 & N_4N_3 & N_4N_4 \end{bmatrix} dV \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}
$$

$$
+ V_x \int_V \begin{bmatrix} N_1 \frac{\partial N_1}{\partial X} & N_1 \frac{\partial N_2}{\partial X} & N_1 \frac{\partial N_3}{\partial X} & N_1 \frac{\partial N_4}{\partial X} \\ N_2 \frac{\partial N_1}{\partial X} & N_2 \frac{\partial N_2}{\partial X} & N_2 \frac{\partial N_3}{\partial X} & N_2 \frac{\partial N_4}{\partial X} \\ N_3 \frac{\partial N_1}{\partial X} & N_3 \frac{\partial N_2}{\partial X} & N_3 \frac{\partial N_3}{\partial X} & N_3 \frac{\partial N_4}{\partial X} \\ N_4 \frac{\partial N_1}{\partial X} & N_4 \frac{\partial N_2}{\partial X} & N_4 \frac{\partial N_3}{\partial X} & N_4 \frac{\partial N_4}{\partial X} \end{bmatrix} dV \{V_x\}^{\Delta\tau+\tau}
$$

$$
+ V_y \int_V \begin{bmatrix} N_1 \frac{\partial N_1}{\partial Y} & N_1 \frac{\partial N_2}{\partial Y} & N_1 \frac{\partial N_3}{\partial Y} & N_1 \frac{\partial N_4}{\partial Y} \\ N_2 \frac{\partial N_1}{\partial Y} & N_2 \frac{\partial N_2}{\partial Y} & N_2 \frac{\partial N_3}{\partial Y} & N_2 \frac{\partial N_4}{\partial Y} \\ N_3 \frac{\partial N_1}{\partial Y} & N_3 \frac{\partial N_2}{\partial Y} & N_3 \frac{\partial N_3}{\partial Y} & N_3 \frac{\partial N_4}{\partial Y} \\ N_4 \frac{\partial N_1}{\partial Y} & N_4 \frac{\partial N_2}{\partial Y} & N_4 \frac{\partial N_3}{\partial Y} & N_4 \frac{\partial N_4}{\partial Y} \end{bmatrix} dV \{V_y\}^{\Delta\tau+\tau}
$$

$$
+ V_z \int_V \begin{bmatrix} N_1 \frac{\partial N_1}{\partial Z} & N_1 \frac{\partial N_2}{\partial Z} & N_1 \frac{\partial N_3}{\partial Z} & N_1 \frac{\partial N_4}{\partial Z} \\ N_2 \frac{\partial N_1}{\partial Z} & N_2 \frac{\partial N_2}{\partial Z} & N_2 \frac{\partial N_3}{\partial Z} & N_2 \frac{\partial N_4}{\partial Z} \\ N_3 \frac{\partial N_1}{\partial Z} & N_3 \frac{\partial N_2}{\partial Z} & N_3 \frac{\partial N_3}{\partial Z} & N_3 \frac{\partial N_4}{\partial Z} \\ N_4 \frac{\partial N_1}{\partial Z} & N_4 \frac{\partial N_2}{\partial Z} & N_4 \frac{\partial N_3}{\partial Z} & N_4 \frac{\partial N_4}{\partial Z} \end{bmatrix} dV \{V_z\}^{\Delta\tau+\tau}
$$

$$+\frac{1}{Ma^2}\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} & N_1\frac{\partial N_4}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} & N_2\frac{\partial N_4}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} & N_3\frac{\partial N_4}{\partial X} \\ N_4\frac{\partial N_1}{\partial X} & N_4\frac{\partial N_2}{\partial X} & N_4\frac{\partial N_3}{\partial X} & N_4\frac{\partial N_4}{\partial X} \end{bmatrix} dV\{V_x\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} & N_1\frac{\partial N_4}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} & N_2\frac{\partial N_4}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} & N_3\frac{\partial N_4}{\partial Y} \\ N_4\frac{\partial N_1}{\partial Y} & N_4\frac{\partial N_2}{\partial Y} & N_4\frac{\partial N_3}{\partial Y} & N_4\frac{\partial N_4}{\partial Y} \end{bmatrix} dV\{V_y\}^{\Delta\tau+\tau}$$

$$+\frac{1}{Ma^2}\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Z} & N_1\frac{\partial N_2}{\partial Z} & N_1\frac{\partial N_3}{\partial Z} & N_1\frac{\partial N_4}{\partial Z} \\ N_2\frac{\partial N_1}{\partial Z} & N_2\frac{\partial N_2}{\partial Z} & N_2\frac{\partial N_3}{\partial Z} & N_2\frac{\partial N_4}{\partial Z} \\ N_3\frac{\partial N_1}{\partial Z} & N_3\frac{\partial N_2}{\partial Z} & N_3\frac{\partial N_3}{\partial Z} & N_3\frac{\partial N_4}{\partial Z} \\ N_4\frac{\partial N_1}{\partial Z} & N_4\frac{\partial N_2}{\partial Z} & N_4\frac{\partial N_3}{\partial Z} & N_4\frac{\partial N_4}{\partial Z} \end{bmatrix} dV\{V_z\}^{\Delta\tau+\tau}$$

$$
=\int_V \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2L_2 & L_2L_3 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3L_3 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4L_4 \end{bmatrix} dV \frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}
$$

$$
+V_x\int_V \begin{bmatrix} L_1\frac{\partial L_1}{\partial X} & L_1\frac{\partial L_2}{\partial X} & L_1\frac{\partial L_3}{\partial X} & L_1\frac{\partial L_4}{\partial X} \\ L_2\frac{\partial L_1}{\partial X} & L_2\frac{\partial L_2}{\partial X} & L_2\frac{\partial L_3}{\partial X} & L_2\frac{\partial L_4}{\partial X} \\ L_3\frac{\partial L_1}{\partial X} & L_3\frac{\partial L_2}{\partial X} & L_3\frac{\partial L_3}{\partial X} & L_3\frac{\partial L_4}{\partial X} \\ L_4\frac{\partial L_1}{\partial X} & L_4\frac{\partial L_2}{\partial X} & L_4\frac{\partial L_3}{\partial X} & L_4\frac{\partial L_4}{\partial X} \end{bmatrix} dV\{P\}^{\Delta\tau+\tau}
$$

$$
+V_y\int_V \begin{bmatrix} L_1\frac{\partial L_1}{\partial Y} & L_1\frac{\partial L_2}{\partial Y} & L_1\frac{\partial L_3}{\partial Y} & L_1\frac{\partial L_4}{\partial Y} \\ L_2\frac{\partial L_1}{\partial Y} & L_2\frac{\partial L_2}{\partial Y} & L_2\frac{\partial L_3}{\partial Y} & L_2\frac{\partial L_4}{\partial Y} \\ L_3\frac{\partial L_1}{\partial Y} & L_3\frac{\partial L_2}{\partial Y} & L_3\frac{\partial L_3}{\partial Y} & L_3\frac{\partial L_4}{\partial Y} \\ L_4\frac{\partial L_1}{\partial Y} & L_4\frac{\partial L_2}{\partial Y} & L_4\frac{\partial L_3}{\partial Y} & L_4\frac{\partial L_4}{\partial Y} \end{bmatrix} dV\{P\}^{\Delta\tau+\tau}
$$

$$
+V_z\int_V \begin{bmatrix} L_1\frac{\partial L_1}{\partial Z} & L_1\frac{\partial L_2}{\partial Z} & L_1\frac{\partial L_3}{\partial Z} & L_1\frac{\partial L_4}{\partial Z} \\ L_2\frac{\partial L_1}{\partial Z} & L_2\frac{\partial L_2}{\partial Z} & L_2\frac{\partial L_3}{\partial Z} & L_2\frac{\partial L_4}{\partial Z} \\ L_3\frac{\partial L_1}{\partial Z} & L_3\frac{\partial L_2}{\partial Z} & L_3\frac{\partial L_3}{\partial Z} & L_3\frac{\partial L_4}{\partial Z} \\ L_4\frac{\partial L_1}{\partial Z} & L_4\frac{\partial L_2}{\partial Z} & L_4\frac{\partial L_3}{\partial Z} & L_4\frac{\partial L_4}{\partial Z} \end{bmatrix} dV\{P\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2}\int_V \begin{bmatrix} L_1\frac{\partial L_1}{\partial X} & L_1\frac{\partial L_2}{\partial X} & L_1\frac{\partial L_3}{\partial X} & L_1\frac{\partial L_4}{\partial X} \\ L_2\frac{\partial L_1}{\partial X} & L_2\frac{\partial L_2}{\partial X} & L_2\frac{\partial L_3}{\partial X} & L_2\frac{\partial L_4}{\partial X} \\ L_3\frac{\partial L_1}{\partial X} & L_3\frac{\partial L_2}{\partial X} & L_3\frac{\partial L_3}{\partial X} & L_3\frac{\partial L_4}{\partial X} \\ L_4\frac{\partial L_1}{\partial X} & L_4\frac{\partial L_2}{\partial X} & L_4\frac{\partial L_3}{\partial X} & L_4\frac{\partial L_4}{\partial X} \end{bmatrix} dV\{V_x\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2}\int_V \begin{bmatrix} L_1\frac{\partial L_1}{\partial Y} & L_1\frac{\partial L_2}{\partial Y} & L_1\frac{\partial L_3}{\partial Y} & L_1\frac{\partial L_4}{\partial Y} \\ L_2\frac{\partial L_1}{\partial Y} & L_2\frac{\partial L_2}{\partial Y} & L_2\frac{\partial L_3}{\partial Y} & L_2\frac{\partial L_4}{\partial Y} \\ L_3\frac{\partial L_1}{\partial Y} & L_3\frac{\partial L_2}{\partial Y} & L_3\frac{\partial L_3}{\partial Y} & L_3\frac{\partial L_4}{\partial Y} \\ L_4\frac{\partial L_1}{\partial Y} & L_4\frac{\partial L_2}{\partial Y} & L_4\frac{\partial L_3}{\partial Y} & L_4\frac{\partial L_4}{\partial Y} \end{bmatrix} dV\{V_y\}^{\Delta\tau+\tau}
$$

$$
= \int_V \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2L_2 & L_2L_3 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3L_3 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4L_4 \end{bmatrix} dV \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}
$$

$$
+ V_x \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} & L_1c_{4x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} & L_2c_{4x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} & L_3c_{4x} \\ L_4c_{1x} & L_4c_{2x} & L_4c_{3x} & L_4c_{4x} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}
$$

$$
+ V_y \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} & L_1c_{4y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} & L_2c_{4y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} & L_3c_{4y} \\ L_4c_{1y} & L_4c_{2y} & L_4c_{3y} & L_4c_{4y} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}
$$

$$
+ V_z \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1z} & L_1c_{2z} & L_1c_{3z} & L_1c_{4z} \\ L_2c_{1z} & L_2c_{2z} & L_2c_{3z} & L_2c_{4z} \\ L_3c_{1z} & L_3c_{2z} & L_3c_{3z} & L_3c_{4z} \\ L_4c_{1z} & L_4c_{2z} & L_4c_{3z} & L_4c_{4z} \end{bmatrix} dV \{P\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2} \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} & L_1c_{4x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} & L_2c_{4x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} & L_3c_{4x} \\ L_4c_{1x} & L_4c_{2x} & L_4c_{3x} & L_4c_{4x} \end{bmatrix} dV \{V_x\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2} \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} & L_1c_{4y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} & L_2c_{4y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} & L_3c_{4y} \\ L_4c_{1y} & L_4c_{2y} & L_4c_{3y} & L_4c_{4y} \end{bmatrix} dV \{V_y\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2} \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1z} & L_1c_{2z} & L_1c_{3z} & L_1c_{4z} \\ L_2c_{1z} & L_2c_{2z} & L_2c_{3z} & L_2c_{4z} \\ L_3c_{1z} & L_3c_{2z} & L_3c_{3z} & L_3c_{4z} \\ L_4c_{1z} & L_4c_{2z} & L_4c_{3z} & L_4c_{4z} \end{bmatrix} dV \{V_z\}^{\Delta\tau+\tau}
$$

ここで、

$$\int_V L_1^p L_2^q L_3^r L_4^s dV = \frac{p!q!r!s!}{(p+q+r+s+3)!} 6V$$

$$\int_V L_i L_j dV = \begin{cases} \dfrac{1!1!}{(1+1+3)!} 6V = \dfrac{6}{5!} V = \dfrac{1}{20} V & (i \neq j) \\ \dfrac{2!}{(1+1+3)!} 6V = \dfrac{12}{5!} V = \dfrac{1}{10} V & (i = j) \end{cases}$$

$$\int_V L_i dV = \frac{1!}{(1+3)!} 6V = \frac{6}{4!} V = \frac{1}{4} V$$

なので、

$$
= \frac{V}{20}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}\frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}
$$

$$
+\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix}\{P\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix}\{P\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix}\{P\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2}\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix}\{V_x\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2}\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix}\{V_y\}^{\Delta\tau+\tau}
$$

$$
+\frac{1}{Ma^2}\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix}\{V_z\}^{\Delta\tau+\tau}
$$

$$
= \frac{V}{20}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \frac{\{P\}^{\Delta\tau+\tau} - \{P\}^{\tau}}{\Delta\tau}
$$

$$
+ \frac{1}{24}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix} \{P\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{24}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix} \{P\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{24}\begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix} \{P\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2}\frac{1}{24}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix} \{V_x\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2}\frac{1}{24}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix} \{V_y\}^{\Delta\tau+\tau}
$$

$$
+ \frac{1}{Ma^2}\frac{1}{24}\begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix} \{V_z\}^{\Delta\tau+\tau}
$$

$$=[C]\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}$$
$$+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}+V_z[C_Z]\{P\}^{\Delta\tau+\tau}$$
$$+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_Z]\{V_z\}^{\Delta\tau+\tau}$$

$$=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$[C]\frac{\{P\}^{\Delta\tau+\tau}-\{P\}^{\tau}}{\Delta\tau}+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}+V_z[C_Z]\{P\}^{\Delta\tau+\tau}$$
$$+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_Z]\{V_z\}^{\Delta\tau+\tau}=0$$

$$\frac{[C]}{\Delta\tau}\{P\}^{\Delta\tau+\tau}$$
$$+V_x[C_x]\{P\}^{\Delta\tau+\tau}+V_y[C_y]\{P\}^{\Delta\tau+\tau}+V_z[C_Z]\{P\}^{\Delta\tau+\tau}$$
$$+\frac{1}{Ma^2}[C_x]\{V_x\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_y]\{V_y\}^{\Delta\tau+\tau}+\frac{1}{Ma^2}[C_Z]\{V_z\}^{\Delta\tau+\tau}=\frac{[C]}{\Delta\tau}\{P\}^{\tau}$$

$$(\frac{[C]}{\Delta\tau}+V_x[C_x]+V_y[C_y]+V_z[C_Z])\{P\}^{\Delta\tau+\tau}$$
$$+\frac{1}{Ma^2}([C_x]\{V_x\}^{\Delta\tau+\tau}+[C_y]\{V_y\}^{\Delta\tau+\tau}+[C_Z]\{V_z\}^{\Delta\tau+\tau})=\frac{[C]}{\Delta\tau}\{P\}^{\tau}$$

・運動量収支式

$$\phi_x = \frac{\partial V_x}{\partial \tau} + V_x \frac{\partial V_x}{\partial X} + V_y \frac{\partial V_x}{\partial Y} + V_z \frac{\partial V_x}{\partial Z} - \frac{\partial \sigma^*_{xx}}{\partial X} - \frac{\partial \sigma^*_{yx}}{\partial Y} - \frac{\partial \sigma^*_{zx}}{\partial Z} - g^*_x = 0$$

$$\phi_y = \frac{\partial V_y}{\partial \tau} + V_x \frac{\partial V_y}{\partial X} + V_y \frac{\partial V_y}{\partial Y} + V_z \frac{\partial V_y}{\partial Z} - \frac{\partial \sigma^*_{xy}}{\partial X} - \frac{\partial \sigma^*_{yy}}{\partial Y} - \frac{\partial \sigma^*_{zy}}{\partial Z} - g^*_y = 0$$

$$\phi_z = \frac{\partial V_z}{\partial \tau} + V_x \frac{\partial V_z}{\partial X} + V_y \frac{\partial V_z}{\partial Y} + V_z \frac{\partial V_z}{\partial Z} - \frac{\partial \sigma^*_{xz}}{\partial X} - \frac{\partial \sigma^*_{yz}}{\partial Y} - \frac{\partial \sigma^*_{zz}}{\partial Z} - g^*_z = 0$$

要素に 4 面体を用いた場合、

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \phi_i dV = \int_V \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dV = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x,1} + N_2 V_{x,2} + N_3 V_{x,3} + N_4 V_{x,4} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} V_{x,1} \\ V_{x,2} \\ V_{x,3} \\ V_{x,4} \end{Bmatrix} = [N]\{V_x\}$$

$$V_y = N_1 V_{y,1} + N_2 V_{y,2} + N_3 V_{y,3} + N_4 V_{y,4}$$
$$V_z = N_1 V_{z,1} + N_2 V_{z,2} + N_3 V_{z,3} + N_4 V_{z,4}$$
$$P = N_1 P_{z,1} + N_2 P_{z,2} + N_3 P_{z,3} + N_4 P_{z,4}$$

となるので、

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left(\frac{\partial V_i}{\partial \tau} + V_x \frac{\partial V_i}{\partial X} + V_y \frac{\partial V_i}{\partial Y} + V_z \frac{\partial V_i}{\partial Z} - \frac{\partial \sigma^*_{xi}}{\partial X} - \frac{\partial \sigma^*_{yi}}{\partial Y} - \frac{\partial \sigma^*_{zi}}{\partial Z} - g^*_i\right) dV$$

$$= \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial \tau} dV + V_x \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial X} dV + V_y \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial Y} dV + V_Z \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial Z} dV$$

$$- \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma^*_{xi}}{\partial X} dV - \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma^*_{yi}}{\partial Y} dV - \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma^*_{zi}}{\partial Z} dV - \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} g^*_i dV$$

展開すると

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left(\frac{\partial V_i}{\partial \tau} + V_x \frac{\partial V_i}{\partial X} + V_y \frac{\partial V_i}{\partial Y} + V_z \frac{\partial V_i}{\partial Z} - \frac{\partial \sigma_{xi}^*}{\partial X} - \frac{\partial \sigma_{yi}^*}{\partial Y} - \frac{\partial \sigma_{zi}^*}{\partial Z} - g_i^*\right) dV$$

$$= \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial \tau} dV$$

$$+ V_x \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial X} dV + V_y \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial Y} dV + V_Z \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial V_i}{\partial Z} dV$$

$$- \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma_{xi}^*}{\partial X} dV - \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma_{yi}^*}{\partial Y} dV - \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \sigma_{zi}^*}{\partial Z} dV$$

$$- \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} g_i^* dV$$

$$= \int_V [N]^T \frac{\partial V_i}{\partial \tau} dV$$

$$+ V_x \int_V [N]^T \frac{\partial V_i}{\partial X} dV + V_y \int_V [N]^T \frac{\partial V_i}{\partial Y} dV + V_Z \int_V [N]^T \frac{\partial V_i}{\partial Z} dV$$

$$- \int_V [N]^T \frac{\partial \sigma_{xi}^*}{\partial X} dV - \int_V [N]^T \frac{\partial \sigma_{yi}^*}{\partial Y} dV - \int_V [N]^T \frac{\partial \sigma_{zi}^*}{\partial Z} dV$$

$$- \int_V [N]^T g_i^* dV$$

グリーン・ガウスの定理より

$$= \int_V [N]^{\mathrm{T}} \frac{[N](\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau})}{\Delta\tau} dV$$

$$+ V_x \int_V [N]^{\mathrm{T}} \frac{\partial[N]\{V_i\}}{\partial X} dV + V_y \int_V [N]^{\mathrm{T}} \frac{\partial[N]\{V_i\}}{\partial Y} dV + V_Z \int_V [N]^{\mathrm{T}} \frac{\partial[N]\{V_i\}}{\partial Z} dV$$

$$- \int_S [N]^{\mathrm{T}} \sigma_{xi}^* n_x dS + \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \sigma_{xi}^* dV$$

$$- \int_S [N]^{\mathrm{T}} \sigma_{yi}^* n_y dS + \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \sigma_{yi}^* dV$$

$$- \int_S [N]^{\mathrm{T}} \sigma_{zi}^* n_z dS + \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \sigma_{zi}^* dV$$

$$- \int_V [N]^{\mathrm{T}} g_i^* dV$$

$$= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\}$$

$$+ \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \sigma_{xi}^* dV + \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \sigma_{yi}^* dV + \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \sigma_{zi}^* dV$$

$$- \int_S [N]^{\mathrm{T}} (\sigma_{xi}^* n_x + \sigma_{yi}^* n_y + \sigma_{zi}^* n_z) dS$$

$$- \int_V [N]^{\mathrm{T}} g_i^* dV$$

$$= \int_V [N]^{\mathrm{T}}[N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\}$$

$$+ \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \{-\delta_{xi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial X} + \frac{\partial V_x}{\partial X_i})\} dV$$

$$+ \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \{-\delta_{yi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial Y} + \frac{\partial V_y}{\partial X_i})\} dV$$

$$+ \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \{-\delta_{zi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial Z} + \frac{\partial V_z}{\partial X_i})\} dV$$

$$- \frac{-2K^*}{We} n_i \int_S [N]^{\mathrm{T}} dS$$

$$- \int_V [N]^{\mathrm{T}} g_i^* dV$$

$$= \int_V [N]^{\mathrm{T}}[N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\}$$

$$+ \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \{-\delta_{xi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial X} + \frac{\partial V_x}{\partial X_i})\} dV$$

$$+ \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \{-\delta_{yi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial Y} + \frac{\partial V_y}{\partial X_i})\} dV$$

$$+ \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \{-\delta_{zi} P + \frac{1}{Re}(\frac{\partial V_i}{\partial Z} + \frac{\partial V_z}{\partial X_i})\} dV$$

$$- \frac{-2K^*}{We} n_i \int_S [N]^{\mathrm{T}} dS$$

$$- \int_V [N]^{\mathrm{T}} g_i^* dV$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\} \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial V_i}{\partial X} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial V_x}{\partial X_i} dV - \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \delta_{xi} P dV \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial V_i}{\partial Y} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial V_y}{\partial X_i} dV - \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \delta_{yi} P dV \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial V_i}{\partial Z} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial V_z}{\partial X_i} dV - \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \delta_{zi} P dV \\
&+ \frac{2K^*}{We} n_i \int_S [N]^{\mathrm{T}} dS \\
&- g_i^* \int_V [N]^{\mathrm{T}} dV
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\} \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial X} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{V_x\}}{\partial X_i} dV - \delta_{xi} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} [N]\{P\} dV \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial Y} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{V_y\}}{\partial X_i} dV - \delta_{yi} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} [N]\{P\} dV \\
&+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial [N]\{V_i\}}{\partial Z} dV + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial [N]\{V_z\}}{\partial X_i} dV - \delta_{zi} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} [N]\{P\} dV \\
&+ \frac{2K^*}{We} n_i \int_S [N]^{\mathrm{T}} dS \\
&- g_i^* \int_V [N]^{\mathrm{T}} dV
\end{aligned}
$$

$$= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + V_Z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\}$$

$$+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{V_i\} + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X_i}\right] dV \{V_x\} - \delta_{xi} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} [N] dV \{P\}$$

$$+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{V_i\} + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X_i}\right] dV \{V_y\} - \delta_{yi} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} [N] dV \{P\}$$

$$+ \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{V_i\} + \frac{1}{Re} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X_i}\right] dV \{V_z\} - \delta_{zi} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} [N] dV \{P\}$$

$$+ \frac{2K^*}{We} n_i \int_S [N]^{\mathrm{T}} dS$$

$$- g_i^* \int_V [N]^{\mathrm{T}} dV \qquad (i = 1,2,3)$$

蓄積量

$$= \int_V \begin{bmatrix} N_1 N_1 & N_1 N_2 & N_1 N_3 & N_1 N_4 \\ N_2 N_1 & N_2 N_2 & N_2 N_3 & N_2 N_4 \\ N_3 N_1 & N_3 N_2 & N_3 N_3 & N_3 N_4 \\ N_4 N_1 & N_4 N_2 & N_4 N_3 & N_4 N_4 \end{bmatrix} dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

対流項

$$+V_x\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} & N_1\frac{\partial N_4}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} & N_2\frac{\partial N_4}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} & N_3\frac{\partial N_4}{\partial X} \\ N_4\frac{\partial N_1}{\partial X} & N_4\frac{\partial N_2}{\partial X} & N_4\frac{\partial N_3}{\partial X} & N_4\frac{\partial N_4}{\partial X} \end{bmatrix} dV\left\{V_i\right\}$$

$$+V_y\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} & N_1\frac{\partial N_4}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} & N_2\frac{\partial N_4}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} & N_3\frac{\partial N_4}{\partial Y} \\ N_4\frac{\partial N_1}{\partial Y} & N_4\frac{\partial N_2}{\partial Y} & N_4\frac{\partial N_3}{\partial Y} & N_4\frac{\partial N_4}{\partial Y} \end{bmatrix} dV\left\{V_i\right\}$$

$$+V_Z\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Z} & N_1\frac{\partial N_2}{\partial Z} & N_1\frac{\partial N_3}{\partial Z} & N_1\frac{\partial N_4}{\partial Z} \\ N_2\frac{\partial N_1}{\partial Z} & N_2\frac{\partial N_2}{\partial Z} & N_2\frac{\partial N_3}{\partial Z} & N_2\frac{\partial N_4}{\partial Z} \\ N_3\frac{\partial N_1}{\partial Z} & N_3\frac{\partial N_2}{\partial Z} & N_3\frac{\partial N_3}{\partial Z} & N_3\frac{\partial N_4}{\partial Z} \\ N_4\frac{\partial N_1}{\partial Z} & N_4\frac{\partial N_2}{\partial Z} & N_4\frac{\partial N_3}{\partial Z} & N_4\frac{\partial N_4}{\partial Z} \end{bmatrix} dV\left\{V_i\right\}$$

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_4}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_4}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_4}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_4}{\partial X}\frac{\partial N_4}{\partial X} \end{bmatrix} dV\{V_i\}$$

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_1}{\partial X}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_2}{\partial X}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_3}{\partial X}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_4}{\partial X}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_4}{\partial X}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_4}{\partial X}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_4}{\partial X}\frac{\partial N_4}{\partial X_i} \end{bmatrix} dV\{V_x\}$$

$$-\delta_{xi}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial X}N_1 & \frac{\partial N_1}{\partial X}N_2 & \frac{\partial N_1}{\partial X}N_3 & \frac{\partial N_1}{\partial X}N_4 \\ \frac{\partial N_2}{\partial X}N_1 & \frac{\partial N_2}{\partial X}N_2 & \frac{\partial N_2}{\partial X}N_3 & \frac{\partial N_2}{\partial X}N_4 \\ \frac{\partial N_3}{\partial X}N_1 & \frac{\partial N_3}{\partial X}N_2 & \frac{\partial N_3}{\partial X}N_3 & \frac{\partial N_3}{\partial X}N_4 \\ \frac{\partial N_4}{\partial X}N_1 & \frac{\partial N_4}{\partial X}N_2 & \frac{\partial N_4}{\partial X}N_3 & \frac{\partial N_4}{\partial X}N_4 \end{bmatrix} dV\{P\}$$

圧力項の x 成分

粘性項の y 成分

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_4}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_4}{\partial Y} \end{bmatrix} dV\left\{V_i\right\}$$

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_1}{\partial Y}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_2}{\partial Y}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_3}{\partial Y}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_4}{\partial Y}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_4}{\partial Y}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_4}{\partial Y}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_4}{\partial Y}\frac{\partial N_4}{\partial X_i} \end{bmatrix} dV\left\{V_y\right\}$$

圧力項の y 成分

$$-\delta_{yi}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Y}N_1 & \frac{\partial N_1}{\partial Y}N_2 & \frac{\partial N_1}{\partial Y}N_3 & \frac{\partial N_1}{\partial Y}N_4 \\ \frac{\partial N_2}{\partial Y}N_1 & \frac{\partial N_2}{\partial Y}N_2 & \frac{\partial N_2}{\partial Y}N_3 & \frac{\partial N_2}{\partial Y}N_4 \\ \frac{\partial N_3}{\partial Y}N_1 & \frac{\partial N_3}{\partial Y}N_2 & \frac{\partial N_3}{\partial Y}N_3 & \frac{\partial N_3}{\partial Y}N_4 \\ \frac{\partial N_4}{\partial Y}N_1 & \frac{\partial N_4}{\partial Y}N_2 & \frac{\partial N_4}{\partial Y}N_3 & \frac{\partial N_4}{\partial Y}N_4 \end{bmatrix} dV\left\{P\right\}$$

粘性項の z 成分

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_2}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_3}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_4}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_4}{\partial Z} \end{bmatrix} dV\{V_i\}$$

$$+\frac{1}{Re}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Z}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_1}{\partial Z}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_1}{\partial Z}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_1}{\partial Z}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_2}{\partial Z}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_2}{\partial Z}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_2}{\partial Z}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_2}{\partial Z}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_3}{\partial Z}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_3}{\partial Z}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_3}{\partial Z}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_3}{\partial Z}\frac{\partial N_4}{\partial X_i} \\ \frac{\partial N_4}{\partial Z}\frac{\partial N_1}{\partial X_i} & \frac{\partial N_4}{\partial Z}\frac{\partial N_2}{\partial X_i} & \frac{\partial N_4}{\partial Z}\frac{\partial N_3}{\partial X_i} & \frac{\partial N_4}{\partial Z}\frac{\partial N_4}{\partial X_i} \end{bmatrix} dV\{V_z\}$$

圧力項の z 成分

$$-\delta_{zi}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Z}N_1 & \frac{\partial N_1}{\partial Z}N_2 & \frac{\partial N_1}{\partial Z}N_3 & \frac{\partial N_1}{\partial Z}N_4 \\ \frac{\partial N_2}{\partial Z}N_1 & \frac{\partial N_2}{\partial Z}N_2 & \frac{\partial N_2}{\partial Z}N_3 & \frac{\partial N_2}{\partial Z}N_4 \\ \frac{\partial N_3}{\partial Z}N_1 & \frac{\partial N_3}{\partial Z}N_2 & \frac{\partial N_3}{\partial Z}N_3 & \frac{\partial N_3}{\partial Z}N_4 \\ \frac{\partial N_4}{\partial Z}N_1 & \frac{\partial N_4}{\partial Z}N_2 & \frac{\partial N_4}{\partial Z}N_3 & \frac{\partial N_4}{\partial Z}N_4 \end{bmatrix} dV\{P\}$$

表面張力項、重力項

$$+\frac{2K^*}{We}n_i\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dS$$

$$-g_i^*\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} dV \qquad (i=1,2,3)$$

$$= \int_V \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2L_2 & L_2L_3 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3L_3 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4L_4 \end{bmatrix} dV \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} & L_1c_{4x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} & L_2c_{4x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} & L_3c_{4x} \\ L_4c_{1x} & L_4c_{2x} & L_4c_{3x} & L_4c_{4x} \end{bmatrix} dV \{V_i\}$$

$$+ V_y \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} & L_1c_{4y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} & L_2c_{4y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} & L_3c_{4y} \\ L_4c_{1y} & L_4c_{2y} & L_4c_{3y} & L_4c_{4y} \end{bmatrix} dV \{V_i\}$$

$$+ V_Z \int_V \frac{1}{6V} \begin{bmatrix} L_1c_{1z} & L_1c_{2z} & L_1c_{3z} & L_1c_{4z} \\ L_2c_{1z} & L_2c_{2z} & L_2c_{3z} & L_2c_{4z} \\ L_3c_{1z} & L_3c_{2z} & L_3c_{3z} & L_3c_{4z} \\ L_4c_{1z} & L_4c_{2z} & L_4c_{3z} & L_4c_{4z} \end{bmatrix} dV \{V_i\}$$

$$+ \frac{1}{Re} \int_V \frac{1}{36V^2} \begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} & c_{1x}c_{4x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} & c_{2x}c_{4x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} & c_{3x}c_{4x} \\ c_{4x}c_{1x} & c_{4x}c_{2x} & c_{4x}c_{3x} & c_{4x}c_{4x} \end{bmatrix} dV \{V_i\}$$

$$+ \frac{1}{Re} \int_V \frac{1}{36V^2} \begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} & c_{1x}c_{4i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} & c_{2x}c_{4i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} & c_{3x}c_{4i} \\ c_{4x}c_{1i} & c_{4x}c_{2i} & c_{4x}c_{3i} & c_{4x}c_{4i} \end{bmatrix} dV \{V_x\}$$

$$- \delta_{xi} \int_V \frac{1}{6V} \begin{bmatrix} c_{1x}L_1 & c_{1x}L_2 & c_{1x}L_3 & c_{1x}L_4 \\ c_{2x}L_1 & c_{2x}L_2 & c_{2x}L_3 & c_{2x}L_4 \\ c_{3x}L_1 & c_{3x}L_2 & c_{3x}L_3 & c_{3x}L_4 \\ c_{4x}L_1 & c_{4x}L_2 & c_{4x}L_3 & c_{4x}L_4 \end{bmatrix} dV \{P\}$$

$$+\frac{1}{Re}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} & c_{1y}c_{4y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} & c_{2y}c_{4y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} & c_{3y}c_{4y} \\ c_{4y}c_{1y} & c_{4y}c_{2y} & c_{4y}c_{3y} & c_{4y}c_{4y} \end{bmatrix} dV\{V_i\}$$

$$+\frac{1}{Re}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} & c_{1y}c_{4i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} & c_{2y}c_{4i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} & c_{3y}c_{4i} \\ c_{4y}c_{1i} & c_{4y}c_{2i} & c_{4y}c_{3i} & c_{4y}c_{4i} \end{bmatrix} dV\{V_y\}$$

$$-\delta_{yi}\int_V \frac{1}{6V}\begin{bmatrix} c_{1y}L_1 & c_{1y}L_2 & c_{1y}L_3 & c_{1y}L_4 \\ c_{2y}L_1 & c_{2y}L_2 & c_{2y}L_3 & c_{2y}L_4 \\ c_{3y}L_1 & c_{3y}L_2 & c_{3y}L_3 & c_{3y}L_4 \\ c_{4y}L_1 & c_{4y}L_2 & c_{4y}L_3 & c_{4y}L_4 \end{bmatrix} dV\{P\}$$

$$+\frac{1}{Re}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1z}c_{1z} & c_{1z}c_{2z} & c_{1z}c_{3z} & c_{1z}c_{4z} \\ c_{2z}c_{1z} & c_{2z}c_{2z} & c_{2z}c_{3z} & c_{2z}c_{4z} \\ c_{3z}c_{1z} & c_{3z}c_{2z} & c_{3z}c_{3z} & c_{3z}c_{4z} \\ c_{4z}c_{1z} & c_{4z}c_{2z} & c_{4z}c_{3z} & c_{4z}c_{4z} \end{bmatrix} dV\{V_i\}$$

$$+\frac{1}{Re}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1z}c_{1i} & c_{1z}c_{2i} & c_{1z}c_{3i} & c_{1z}c_{4i} \\ c_{2z}c_{1i} & c_{2z}c_{2i} & c_{2z}c_{3i} & c_{2z}c_{4i} \\ c_{3z}c_{1i} & c_{3z}c_{2i} & c_{3z}c_{3i} & c_{3z}c_{4i} \\ c_{4z}c_{1i} & c_{4z}c_{2i} & c_{4z}c_{3i} & c_{4z}c_{4i} \end{bmatrix} dV\{V_z\}$$

$$-\delta_{zi}\int_V \frac{1}{6V}\begin{bmatrix} c_{1z}L_1 & c_{1z}L_2 & c_{1z}L_3 & c_{1z}L_4 \\ c_{2z}L_1 & c_{2z}L_2 & c_{2z}L_3 & c_{2z}L_4 \\ c_{3z}L_1 & c_{3z}L_2 & c_{3z}L_3 & c_{3z}L_4 \\ c_{4z}L_1 & c_{4z}L_2 & c_{4z}L_3 & c_{4z}L_4 \end{bmatrix} dV\{P\}$$

$$+\frac{2K^*}{We}n_i\int_S \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} dS$$

$$-g_i^* \int_V \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}^{\mathrm{T}} dV \qquad (i = 1,2,3)$$

ここで、

$$\int_V L_1^p L_2^q L_3^r L_4^s dV = \frac{p!q!r!s!}{(p+q+r+s+3)!} 6V$$

$$\int_V L_i L_j dV = \begin{cases} \dfrac{1!1!}{(1+1+3)!} 6V = \dfrac{6}{5!} V = \dfrac{1}{20} V & (i \neq j) \\ \dfrac{2!}{(1+1+3)!} 6V = \dfrac{12}{5!} V = \dfrac{1}{10} V & (i = j) \end{cases}$$

$$\int_V L_i dV = \frac{1!}{(1+3)!} 6V = \frac{6}{4!} V = \frac{1}{4} V$$

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!} 2S$$

$$\int_S L_i dS = \frac{1}{(1+2)!} 2S = \frac{1}{3} S$$

$$= \frac{1}{20} V \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau}$$

$$+ V_x \frac{1}{6V} \frac{V}{4} \begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix} \{V_i\}$$

$$+ V_y \frac{1}{6V} \frac{V}{4} \begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix} \{V_i\}$$

$$+ V_Z \frac{1}{6V} \frac{V}{4} \begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix} \{V_i\}$$

$$+ \frac{1}{Re} \frac{1}{36V^2} V \begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} & c_{1x}c_{4x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} & c_{2x}c_{4x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} & c_{3x}c_{4x} \\ c_{4x}c_{1x} & c_{4x}c_{2x} & c_{4x}c_{3x} & c_{4x}c_{4x} \end{bmatrix} \{V_i\}$$

$$+ \frac{1}{Re} \frac{1}{36V^2} V \begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} & c_{1x}c_{4i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} & c_{2x}c_{4i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} & c_{3x}c_{4i} \\ c_{4x}c_{1i} & c_{4x}c_{2i} & c_{4x}c_{3i} & c_{4x}c_{4i} \end{bmatrix} \{V_x\}$$

$$- \delta_{xi} \frac{1}{6V} \frac{V}{4} \begin{bmatrix} c_{1x} & c_{1x} & c_{1x} & c_{1x} \\ c_{2x} & c_{2x} & c_{2x} & c_{2x} \\ c_{3x} & c_{3x} & c_{3x} & c_{3x} \\ c_{4x} & c_{4x} & c_{4x} & c_{4x} \end{bmatrix} \{P\}$$

$$+\frac{1}{Re}\frac{1}{36V^2}V\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} & c_{1y}c_{4y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} & c_{2y}c_{4y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} & c_{3y}c_{4y} \\ c_{4y}c_{1y} & c_{4y}c_{2y} & c_{4y}c_{3y} & c_{4y}c_{4y} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V^2}V\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} & c_{1y}c_{4i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} & c_{2y}c_{4i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} & c_{3y}c_{4i} \\ c_{4y}c_{1i} & c_{4y}c_{2i} & c_{4y}c_{3i} & c_{4y}c_{4i} \end{bmatrix}\{V_y\}$$

$$-\delta_{yi}\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1y} & c_{1y} & c_{1y} & c_{1y} \\ c_{2y} & c_{2y} & c_{2y} & c_{2y} \\ c_{3y} & c_{3y} & c_{3y} & c_{3y} \\ c_{4y} & c_{4y} & c_{4y} & c_{4y} \end{bmatrix}\{P\}$$

$$+\frac{1}{Re}\frac{1}{36V^2}V\begin{bmatrix} c_{1z}c_{1z} & c_{1z}c_{2z} & c_{1z}c_{3z} & c_{1z}c_{4z} \\ c_{2z}c_{1z} & c_{2z}c_{2z} & c_{2z}c_{3z} & c_{2z}c_{4z} \\ c_{3z}c_{1z} & c_{3z}c_{2z} & c_{3z}c_{3z} & c_{3z}c_{4z} \\ c_{4z}c_{1z} & c_{4z}c_{2z} & c_{4z}c_{3z} & c_{4z}c_{4z} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V^2}V\begin{bmatrix} c_{1z}c_{1i} & c_{1z}c_{2i} & c_{1z}c_{3i} & c_{1z}c_{4i} \\ c_{2z}c_{1i} & c_{2z}c_{2i} & c_{2z}c_{3i} & c_{2z}c_{4i} \\ c_{3z}c_{1i} & c_{3z}c_{2i} & c_{3z}c_{3i} & c_{3z}c_{4i} \\ c_{4z}c_{1i} & c_{4z}c_{2i} & c_{4z}c_{3i} & c_{4z}c_{4i} \end{bmatrix}\{V_z\}$$

$$-\delta_{zi}\frac{1}{6V}\frac{V}{4}\begin{bmatrix} c_{1z} & c_{1z} & c_{1z} & c_{1z} \\ c_{2z} & c_{2z} & c_{2z} & c_{2z} \\ c_{3z} & c_{3z} & c_{3z} & c_{3z} \\ c_{4z} & c_{4z} & c_{4z} & c_{4z} \end{bmatrix}\{P\}$$

$$+\frac{2K^*}{We}n_i\frac{S}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$-g_i^* \frac{V}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad (i=1,2,3)$$

$$=\frac{1}{20}V\begin{bmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}\frac{\{V_i\}^{\tau+\Delta\tau}-\{V_i\}^{\tau}}{\Delta\tau}$$

$$+V_x\frac{1}{24}\begin{bmatrix}c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\end{bmatrix}\{V_i\}$$

$$+V_y\frac{1}{24}\begin{bmatrix}c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\end{bmatrix}\{V_i\}$$

$$+V_Z\frac{1}{24}\begin{bmatrix}c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} & c_{1x}c_{4x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} & c_{2x}c_{4x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} & c_{3x}c_{4x} \\ c_{4x}c_{1x} & c_{4x}c_{2x} & c_{4x}c_{3x} & c_{4x}c_{4x} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1x}c_{1i} & c_{1x}c_{2i} & c_{1x}c_{3i} & c_{1x}c_{4i} \\ c_{2x}c_{1i} & c_{2x}c_{2i} & c_{2x}c_{3i} & c_{2x}c_{4i} \\ c_{3x}c_{1i} & c_{3x}c_{2i} & c_{3x}c_{3i} & c_{3x}c_{4i} \\ c_{4x}c_{1i} & c_{4x}c_{2i} & c_{4x}c_{3i} & c_{4x}c_{4i} \end{bmatrix}\{V_x\}$$

$$-\delta_{xi}\frac{1}{24}\begin{bmatrix} c_{1x} & c_{1x} & c_{1x} & c_{1x} \\ c_{2x} & c_{2x} & c_{2x} & c_{2x} \\ c_{3x} & c_{3x} & c_{3x} & c_{3x} \\ c_{4x} & c_{4x} & c_{4x} & c_{4x} \end{bmatrix}\{P\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} & c_{1y}c_{4y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} & c_{2y}c_{4y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} & c_{3y}c_{4y} \\ c_{4y}c_{1y} & c_{4y}c_{2y} & c_{4y}c_{3y} & c_{4y}c_{4y} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1y}c_{1i} & c_{1y}c_{2i} & c_{1y}c_{3i} & c_{1y}c_{4i} \\ c_{2y}c_{1i} & c_{2y}c_{2i} & c_{2y}c_{3i} & c_{2y}c_{4i} \\ c_{3y}c_{1i} & c_{3y}c_{2i} & c_{3y}c_{3i} & c_{3y}c_{4i} \\ c_{4y}c_{1i} & c_{4y}c_{2i} & c_{4y}c_{3i} & c_{4y}c_{4i} \end{bmatrix}\{V_y\}$$

$$-\delta_{yi}\frac{1}{24}\begin{bmatrix} c_{1y} & c_{1y} & c_{1y} & c_{1y} \\ c_{2y} & c_{2y} & c_{2y} & c_{2y} \\ c_{3y} & c_{3y} & c_{3y} & c_{3y} \\ c_{4y} & c_{4y} & c_{4y} & c_{4y} \end{bmatrix}\{P\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1z}c_{1z} & c_{1z}c_{2z} & c_{1z}c_{3z} & c_{1z}c_{4z} \\ c_{2z}c_{1z} & c_{2z}c_{2z} & c_{2z}c_{3z} & c_{2z}c_{4z} \\ c_{3z}c_{1z} & c_{3z}c_{2z} & c_{3z}c_{3z} & c_{3z}c_{4z} \\ c_{4z}c_{1z} & c_{4z}c_{2z} & c_{4z}c_{3z} & c_{4z}c_{4z} \end{bmatrix}\{V_i\}$$

$$+\frac{1}{Re}\frac{1}{36V}\begin{bmatrix} c_{1z}c_{1i} & c_{1z}c_{2i} & c_{1z}c_{3i} & c_{1z}c_{4i} \\ c_{2z}c_{1i} & c_{2z}c_{2i} & c_{2z}c_{3i} & c_{2z}c_{4i} \\ c_{3z}c_{1i} & c_{3z}c_{2i} & c_{3z}c_{3i} & c_{3z}c_{4i} \\ c_{4z}c_{1i} & c_{4z}c_{2i} & c_{4z}c_{3i} & c_{4z}c_{4i} \end{bmatrix}\{V_z\}$$

$$-\delta_{zi}\frac{1}{24}\begin{bmatrix} c_{1z} & c_{1z} & c_{1z} & c_{1z} \\ c_{2z} & c_{2z} & c_{2z} & c_{2z} \\ c_{3z} & c_{3z} & c_{3z} & c_{3z} \\ c_{4z} & c_{4z} & c_{4z} & c_{4z} \end{bmatrix}\{P\}$$

$$+\frac{2}{3}\frac{K^*}{We}n_i S\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$-g_i^*\frac{V}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (i=1,2,3)$$

$$
\begin{aligned}
&= [C]\frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} + V_x[C_x]\{V_i\} + V_y[C_y]\{V_i\} + V_Z[C_z]\{V_i\} \\
&+ \frac{1}{Re}[S_{xx}]\{V_i\} + \frac{1}{Re}[S_{xi}]\{V_x\} - \delta_{xi}[H_x]\{P\} \\
&+ \frac{1}{Re}[S_{yy}]\{V_i\} + \frac{1}{Re}[S_{yi}]\{V_y\} - \delta_{yi}[H_y]\{P\} \\
&+ \frac{1}{Re}[S_{zz}]\{V_i\} + \frac{1}{Re}[S_{zi}]\{V_z\} - \delta_{zi}[H_z]\{P\} \\
&+ \frac{2}{3}\frac{K^*}{We} n_i S \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&- g_i^* \frac{V}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (i = 1,2,3)
\end{aligned}
$$

$$
\begin{aligned}
&= [C]\frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} \\
&+ (V_x[C_x] + V_y[C_y] + V_Z[C_z])\{V_i\} \\
&+ \frac{1}{Re}([S_{xx}] + [S_{yy}] + [S_{zz}])\{V_i\} \\
&+ \frac{1}{Re}([S_{xi}]\{V_x\} + [S_{yi}]\{V_y\} + [S_{zi}]\{V_z\}) \\
&- (\delta_{xi}[H_x] + \delta_{yi}[H_y] + \delta_{zi}[H_z])\{P\} \\
&+ \frac{2}{3}\frac{K^*}{We} n_i S \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&- g_i^* \frac{V}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (i = 1,2,3) \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&[C]\frac{\{V_i\}^{\tau+\Delta\tau} - \{V_i\}^{\tau}}{\Delta\tau} + (V_x[C_x] + V_y[C_y] + V_Z[C_z])\{V_i\} \\
&+ \frac{1}{Re}([S_{xx}] + [S_{yy}] + [S_{zz}])\{V_i\} \\
&+ \frac{1}{Re}([S_{xi}]\{V_x\} + [S_{yi}]\{V_y\} + [S_{zi}]\{V_z\}) \\
&- (\delta_{xi}[H_x] + \delta_{yi}[H_y] + \delta_{zi}[H_z])\{P\} \\
&+ \frac{2}{3}\frac{K^*}{We} n_i S \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - g_i^* \frac{V}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (i = 1,2,3)
\end{aligned}
$$

陰解法

$$\frac{[C]}{\Delta\tau}\{V_i\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y]+V_Z[C_z])\{V_i\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}([S_{xx}]+[S_{yy}]+[S_{zz}])\{V_i\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}([S_{xi}]\{V_x\}^{\tau+\Delta\tau}+[S_{yi}]\{V_y\}^{\tau+\Delta\tau}+[S_{zi}]\{V_z\}^{\tau+\Delta\tau})$$
$$-(\delta_{xi}[H_x]+\delta_{yi}[H_y]+\delta_{zi}[H_z])\{P\}^{\tau+\Delta\tau}$$
$$=\frac{[C]}{\Delta\tau}\{V_i\}^{\tau}-\frac{2}{3}\frac{K^*}{We}n_i S\begin{bmatrix}1\\1\\1\end{bmatrix}+g_i^*\frac{V}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\qquad(i=1,2,3)$$

x 成分

$$\frac{[C]}{\Delta\tau}\{V_x\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y]+V_Z[C_z])\{V_x\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}\{(2[S_{xx}]+[S_{yy}]+[S_{zz}])\{V_x\}^{\tau+\Delta\tau}+[S_{yx}]\{V_y\}^{\tau+\Delta\tau}+[S_{zx}]\{V_z\}^{\tau+\Delta\tau}\}-[H_x]\{P\}^{\tau+\Delta\tau}$$
$$=\frac{[C]}{\Delta\tau}\{V_x\}^{\tau}-\frac{2}{3}\frac{K^*}{We}n_x S\begin{bmatrix}1\\1\\1\end{bmatrix}+g_x^*\frac{V}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

y 成分

$$\frac{[C]}{\Delta\tau}\{V_y\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y]+V_Z[C_z])\{V_y\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}\{[S_{xy}]\{V_x\}^{\tau+\Delta\tau}+([S_{xx}]+2[S_{yy}]+[S_{zz}])\{V_y\}^{\tau+\Delta\tau}+[S_{zy}]\{V_z\}^{\tau+\Delta\tau}\}-[H_y]\{P\}^{\tau+\Delta\tau}$$
$$=\frac{[C]}{\Delta\tau}\{V_y\}^{\tau}-\frac{2}{3}\frac{K^*}{We}n_y S\begin{bmatrix}1\\1\\1\end{bmatrix}+g_y^*\frac{V}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

z 成分

$$
\frac{[C]}{\Delta\tau}\{V_z\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y]+V_Z[C_z])\{V_z\}^{\tau+\Delta\tau}
$$

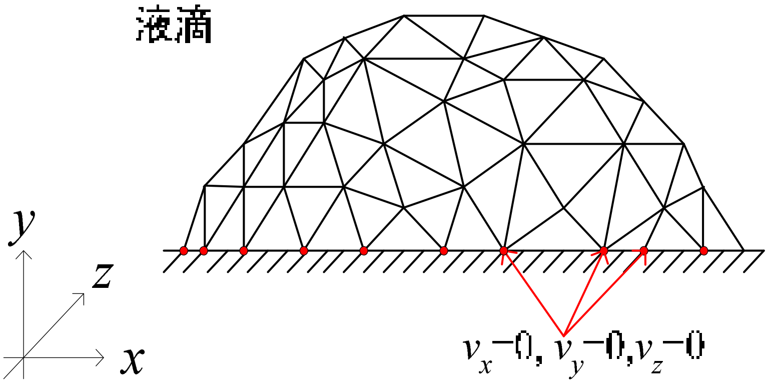
$$
+\frac{1}{Re}\{[S_{xz}]\{V_x\}^{\tau+\Delta\tau}+[S_{yz}]\{V_y\}^{\tau+\Delta\tau}+([S_{xx}]+[S_{yy}]+2[S_{zz}])\{V_z\}^{\tau+\Delta\tau}\}-[H_z]\{P\}^{\tau+\Delta\tau}
$$

$$
=\frac{[C]}{\Delta\tau}\{V_z\}^{\tau}-\frac{2}{3}\frac{K^*}{We}n_zS\begin{bmatrix}1\\1\\1\end{bmatrix}+g_z^*\frac{V}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad (i=1,2,3)
$$

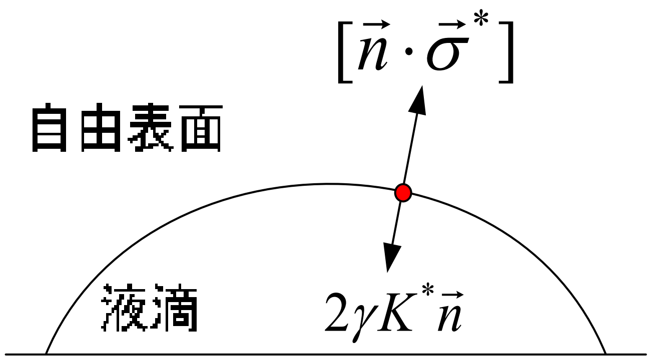
・境界条件
境界条件は
第 1 種境界条件
第 2 種境界条件
第 3 種境界条件
に分けられます。
第 1 種境界条件とは、界面の節点に圧力や速度などの物理量を直接与える条件です。例えば、滑り無しの条件では壁面の速度は 0 になります。
第 2 種境界条件とは、界面の節点に応力[N/m^2]を与える条件です。界面に表面張力が働く場合は、界面節点に応力を加えます。

・第 1 種境界条件

節点に直接、速度、圧力などの物理量を与える条件です。下図は 4 面体 1 次要素の滑り無し条件です。壁面の節点に速度 0 を与えています。



・第2種境界条件
節点に応力 [N/m²]を与える条件です。下図は、界面に表面張力を考慮しています。



$$
\begin{aligned}
[\vec{n}^*\cdot\vec{\sigma}^*] &= [\sum_{i=1}^{3}\vec{\delta}_i n_i \cdot \sum_{j=1}^{3}\sum_{k=1}^{3}\vec{\delta}_j\vec{\delta}_k\sigma^*_{jk}] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}[\vec{\delta}_i\cdot\vec{\delta}_j\vec{\delta}_k]n_i\sigma^*_{jk} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{ij}\vec{\delta}_k n_i\sigma^*_{jk} \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\vec{\delta}_k n_i\sigma^*_{ik} \qquad (\because i=j) \\
&= \sum_{k=1}^{3}\vec{\delta}_k\sum_{i=1}^{3}n_i\sigma^*_{ik} \\
&= \sum_{k=1}^{3}\vec{\delta}_k(n_x\sigma^*_{xk}+n_y\sigma^*_{yk}+n_z\sigma^*_{zk}) \\
&= \vec{\delta}_x(n_x\sigma^*_{xx}+n_y\sigma^*_{yx}+n_z\sigma^*_{zx}) \\
&\quad +\vec{\delta}_y(n_x\sigma^*_{xy}+n_y\sigma^*_{yy}+n_z\sigma^*_{zy}) \\
&\quad +\vec{\delta}_z(n_x\sigma^*_{xz}+n_y\sigma^*_{yz}+n_z\sigma^*_{zz}) \\
&= \vec{\delta}_x\frac{-2K^*}{We}n_x+\vec{\delta}_y\frac{-2K^*}{We}n_y+\vec{\delta}_z\frac{-2K^*}{We}n_z
\end{aligned}
$$

ここで、表面張力は、Laplace の式より

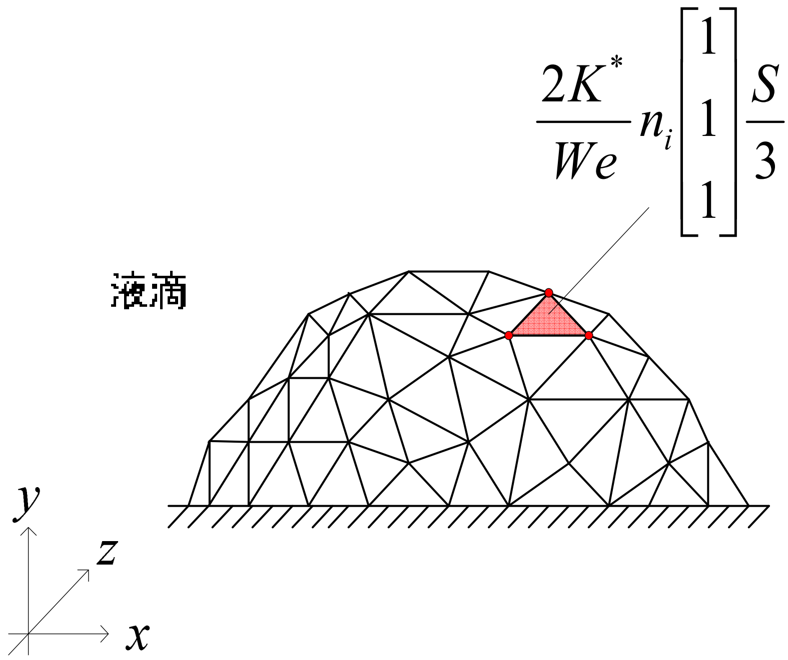
$$[\vec{n}\cdot\vec{\sigma}] = -2\gamma K\vec{n}$$

$$p_{bd} = 2\gamma K$$

なので、無次元化すると

$$
\begin{aligned}
[\vec{n} \cdot \vec{\sigma}^*] &= -\frac{2\gamma K^*}{\rho v_0^2 x_0}\vec{n} \\
&= -\frac{2K^*}{We}\vec{n} \\
&= \vec{\delta}_x \frac{-2K^*}{We} n_x + \vec{\delta}_y \frac{-2K^*}{We} n_y + \vec{\delta}_z \frac{-2K^*}{We} n_z
\end{aligned}
$$

$$
\begin{aligned}
P_{bd}^* &= \frac{P_{bd}}{\rho v_0^2} \\
&= \frac{2\gamma K}{\rho v_0^2} \\
&= \frac{2\gamma K^*}{\rho v_0^2 x_0} \\
&= \frac{2K^*}{We} \\
&= \frac{2K^*}{Oh^2}
\end{aligned}
$$



・座標の取り方

・オイラー座標、ラグラジアン座標

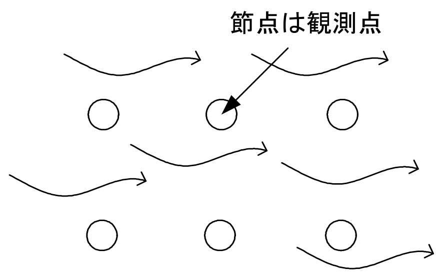
数値計算では、タイムステップごとに節点の速度を計算していきます。節点には、「観測点」という見方と「流体粒子」という見方があり、それぞれオイラー座標、ラグラジアン座標が用いられます。これらの座標の違いは、対流項(慣性力)を考慮するかしないかです。対流項は、全微分作用の項に含まれています。

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}$$

・オイラー座標

節点を観測点と見なして計算します。対流項を考慮するため、節点は移動しません。逆に言えば、節点を固定するため、慣性力によって節点が押されると解釈できます。

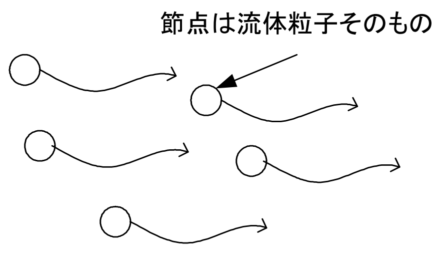
$$\frac{D\vec{v}}{Dt} = [\nabla \cdot \vec{\sigma}] + \rho\vec{g}$$



・ラグラジアン座標

節点を流体粒子と見なして計算します。対流項を考慮しないため、節点は移動します。逆に言えば、慣性力によって節点が押されて移動するため、節点自体は慣性力を感じない(慣性力が蓄積しない)と解釈できます。節点の移動量は、タイムステップΔt[s]×計算した速度 v[m/s]となります。

$$\frac{\partial \vec{v}}{\partial t} = [\nabla \cdot \vec{\sigma}] + \rho\vec{g}$$



座標の取り方には、オイラー座標とラグラジアン座標の 2 つがあります。

・収支式のマトリックス化

数値計算の目的は、解析的に解けない収支式を四則演算に分解し、プログラム化して解くことです。有限要素法もその 1 つです。

質量収支式

$$
\left(\frac{[C]}{5\Delta\tau}+V_x[C_x]+V_y[C_y]+V_z[C_Z]\right)\{P\}^{\Delta\tau+\tau}
$$
$$
+\frac{1}{Ma^2}\left([C_x]\{V_x\}^{\Delta\tau+\tau}+[C_y]\{V_y\}^{\Delta\tau+\tau}+[C_Z]\{V_z\}^{\Delta\tau+\tau}\right)=\frac{[C]}{5\Delta\tau}\{P\}^{\tau}
$$

x 成分

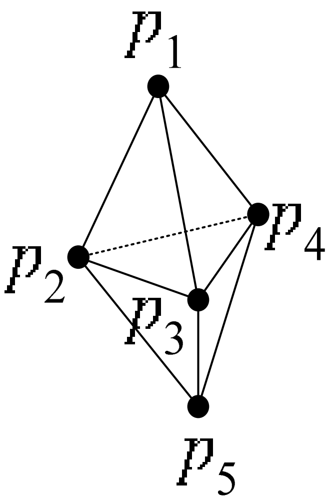
$$
\frac{[C]}{\Delta\tau}\{V_x\}^{\tau+\Delta\tau}+\left(V_x[C_x]+V_y[C_y]+V_Z[C_z]\right)\{V_x\}^{\tau+\Delta\tau}
$$
$$
+\frac{1}{Re}\left\{\left(2[S_{xx}]+[S_{yy}]+[S_{zz}]\right)\{V_x\}^{\tau+\Delta\tau}+[S_{yx}]\{V_y\}^{\tau+\Delta\tau}+[S_{zx}]\{V_z\}^{\tau+\Delta\tau}+[C_x]\{P\}^{\tau+\Delta\tau}\right\}
$$
$$
=\frac{[C]}{\Delta\tau}\{V_x\}^{\tau}+\frac{2K^*}{We}n_x\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{S}{3}+g_x^*\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{V}{4}
$$

y 成分

$$
\frac{[C]}{\Delta\tau}\{V_y\}^{\tau+\Delta\tau}+\left(V_x[C_x]+V_y[C_y]+V_Z[C_z]\right)\{V_y\}^{\tau+\Delta\tau}
$$
$$
+\frac{1}{Re}\left\{[S_{xy}]\{V_x\}^{\tau+\Delta\tau}+\left([S_{xx}]+2[S_{yy}]+[S_{zz}]\right)\{V_y\}^{\tau+\Delta\tau}+[S_{zy}]\{V_z\}^{\tau+\Delta\tau}+[C_y]\{P\}^{\tau+\Delta\tau}\right\}
$$
$$
=\frac{[C]}{\Delta\tau}\{V_y\}^{\tau}+\frac{2K^*}{We}n_y\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{S}{3}+g_y^*\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{V}{4}
$$

z 成分

$$\frac{[C]}{\Delta\tau}\{V_z\}^{\tau+\Delta\tau}+(V_x[C_x]+V_y[C_y]+V_Z[C_z])\{V_z\}^{\tau+\Delta\tau}$$
$$+\frac{1}{Re}\{[S_{xz}]\{V_x\}^{\tau+\Delta\tau}+[S_{yz}]\{V_y\}^{\tau+\Delta\tau}+([S_{xx}]+[S_{yy}]+2[S_{zz}])\{V_z\}^{\tau+\Delta\tau}+[C_z]\{P\}^{\tau+\Delta\tau}\}$$
$$=\frac{[C]}{\Delta\tau}\{V_z\}^{\tau}+\frac{2K^*}{We}n_z\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{S}{3}+g_z^*\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{V}{4} \qquad (i=1,2,3)$$



$p_1(v_{1x}, v_{1y}, v_{1z}, p_1)$

$p_2(v_{2x}, v_{2y}, v_{2z}, p_2)$

$p_3(v_{3x}, v_{3y}, v_{3z}, p_3)$

$p_4(v_{4x}, v_{4y}, v_{4z}, p_4)$

$p_5(v_{5x}, v_{5y}, v_{5z}, p_5)$

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{119} & a_{120} \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{219} & a_{220} \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{319} & a_{320} \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{419} & a_{420} \\
a_{51} & a_{52} & a_{53} & a_{54} & \cdots & a_{519} & a_{520} \\
a_{61} & a_{62} & a_{63} & a_{64} & \cdots & a_{619} & a_{620} \\
a_{71} & a_{72} & a_{73} & a_{74} & \cdots & a_{719} & a_{720} \\
a_{81} & a_{82} & a_{83} & a_{84} & \cdots & a_{819} & a_{820} \\
a_{91} & a_{92} & a_{93} & a_{94} & \cdots & a_{919} & a_{920} \\
a_{101} & a_{102} & a_{103} & a_{104} & \cdots & a_{1019} & a_{1020} \\
a_{111} & a_{112} & a_{113} & a_{114} & \cdots & a_{1119} & a_{1120} \\
a_{121} & a_{122} & a_{123} & a_{124} & \cdots & a_{1219} & a_{1220} \\
a_{131} & a_{132} & a_{133} & a_{134} & \cdots & a_{1319} & a_{1320} \\
a_{141} & a_{142} & a_{143} & a_{144} & \cdots & a_{1419} & a_{1420} \\
a_{151} & a_{152} & a_{153} & a_{154} & \cdots & a_{1519} & a_{1520} \\
a_{161} & a_{162} & a_{163} & a_{164} & \cdots & a_{1619} & a_{1620} \\
a_{171} & a_{172} & a_{173} & a_{174} & \cdots & a_{1719} & a_{1720} \\
a_{181} & a_{182} & a_{183} & a_{184} & \cdots & a_{1819} & a_{1820} \\
a_{191} & a_{192} & a_{193} & a_{194} & \cdots & a_{1919} & a_{1920} \\
a_{201} & a_{202} & a_{203} & a_{204} & \cdots & a_{2019} & a_{2020}
\end{pmatrix}
\begin{pmatrix}
v_{1x} \\ v_{1y} \\ v_{1z} \\ p_1 \\ v_{2x} \\ v_{2y} \\ v_{2z} \\ p_2 \\ v_{3x} \\ v_{3y} \\ v_{3z} \\ p_3 \\ v_{4x} \\ v_{4y} \\ v_{4z} \\ p_4 \\ v_{5x} \\ v_{5y} \\ v_{5z} \\ p_5
\end{pmatrix}^{\tau+\Delta\tau}
=
\begin{pmatrix}
v_{1x} \\ v_{1y} \\ v_{1z} \\ p_1 \\ v_{2x} \\ v_{2y} \\ v_{2z} \\ p_2 \\ v_{3x} \\ v_{3y} \\ v_{3z} \\ p_3 \\ v_{4x} \\ v_{4y} \\ v_{4z} \\ p_4 \\ v_{5x} \\ v_{5y} \\ v_{5z} \\ p_5
\end{pmatrix}^{\tau}
+
\begin{pmatrix}
f_{11} \\ f_{12} \\ f_{13} \\ f_{14} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{31} \\ f_{32} \\ f_{33} \\ f_{34} \\ f_{41} \\ f_{42} \\ f_{43} \\ f_{44} \\ f_{51} \\ f_{52} \\ f_{53} \\ f_{54}
\end{pmatrix}
$$

・行列計算

数値計算を行うためには、多元１次連立方程式を解く必要があります。節点の数にもよるが、数万の連立方程式をタイムステップごとに解く場合もあるので、効率化する必要がります。その方法をいくつか紹介します。解法は、直接法と反復法に分けられます。

直接法　解析的に解く方法。手計算による解法もこちらに含まれます。
反復法　数値計算で、少しずつ解に近づける方法。近似解を計算します。

・ガウスの消去法

ガウスの消去法は、直接、連立方程式を解く方法です。この解法で３元１次方程式を解いてみます。

$$\begin{cases} a_1x_1 + b_1y_1 + c_1z_1 = d_1 \\ a_2x_2 + b_2y_2 + c_2z_2 = d_2 \\ a_3x_3 + b_3y_3 + c_3z_3 = d_3 \end{cases}$$

上式を行列式で表すと次式となります。

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

これを次式の形に直すと変数がわかります。

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

１つの行列で表すと次式となります。

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

手順１

- １行目を係数 a_1 で割る。
- ２行目を係数 a_2 で割る。
- ３行目を係数 a_3 で割る。

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} & \dfrac{d_1}{a_1} \\ 1 & \dfrac{b_2}{a_2} & \dfrac{c_2}{a_2} & \dfrac{d_2}{a_2} \\ 1 & \dfrac{b_3}{a_3} & \dfrac{c_3}{a_3} & \dfrac{d_3}{a_3} \end{pmatrix}$$

手順２　１行目を引く。

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} & \dfrac{d_1}{a_1} \\ 0 & \dfrac{b_2}{a_2}-\dfrac{b_1}{a_1} & \dfrac{c_2}{a_2}-\dfrac{c_1}{a_1} & \dfrac{d_2}{a_2}-\dfrac{d_1}{a_1} \\ 0 & \dfrac{b_3}{a_3}-\dfrac{b_1}{a_1} & \dfrac{c_3}{a_3}-\dfrac{c_1}{a_1} & \dfrac{d_3}{a_3}-\dfrac{d_1}{a_1} \end{pmatrix}$$

手順３　係数を次式で置き換える。

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} & \dfrac{d_1}{a_1} \\ 0 & s_1 & s_2 & s_3 \\ 0 & t_1 & t_2 & t_3 \end{pmatrix}$$

手順４

・２行目を係数 s_1 で割る。

・３行目を係数 t_1 で割る。

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} & \dfrac{d_1}{a_1} \\ 0 & 1 & \dfrac{s_2}{s_1} & \dfrac{s_3}{s_1} \\ 0 & 1 & \dfrac{t_2}{t_1} & \dfrac{t_3}{t_1} \end{pmatrix}$$

手順５　２行目を引く。

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} & \dfrac{d_1}{a_1} \\ 0 & 1 & \dfrac{s_2}{s_1} & \dfrac{s_3}{s_1} \\ 0 & 0 & \dfrac{t_2}{t_1}-\dfrac{s_2}{s_1} & \dfrac{t_3}{t_1}-\dfrac{s_3}{s_1} \end{pmatrix}$$

連立方程式に直すと次式となります。$z \to y \to x$ の順番に変数が計算できます。

$$\begin{cases} x + \dfrac{b_1}{a_1} y + \dfrac{c_1}{a_1} z = \dfrac{d_1}{a_1} \\ y + \dfrac{s_2}{s_1} z = \dfrac{s_3}{s_1} \\ (\dfrac{t_2}{t_1} - \dfrac{s_2}{s_1}) z = \dfrac{t_3}{t_1} - \dfrac{s_3}{s_1} \end{cases}$$

$$\begin{pmatrix} 1 & \dfrac{b_1}{a_1} & \dfrac{c_1}{a_1} \\ 0 & 1 & \dfrac{s_2}{s_1} \\ 0 & 0 & \dfrac{t_2}{t_1} - \dfrac{s_2}{s_1} \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} \dfrac{d_1}{a_1} \\ \dfrac{s_3}{s_1} \\ \dfrac{t_3}{t_1} - \dfrac{s_3}{s_1} \end{pmatrix}$$

・ポイント

割り算をする係数が 0 の場合は、計算ができません。

係数が 0 となる場合は、列を入れ替える必要があります。

割り算をする係数は、a_{ii} 成分です。

・バンド幅の最適化

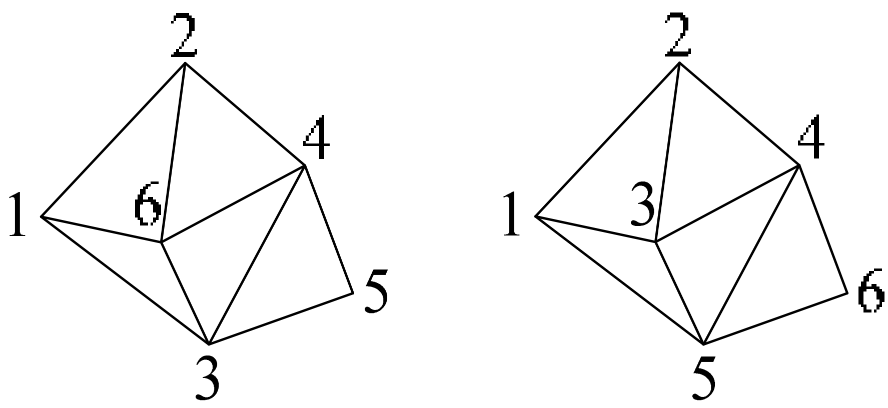
有限要素法では、最終的に数千、数万オーダーの連立１次方程式を解くことになります。
ここで、バンド幅をできるだけ小さくすれば、より計算量を小さくすることができます。
バンド幅は、下図で表されます。

バンド幅

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & 0 \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & 0 & 0 & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ 0 & 0 & 0 & 0 & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \\ \phi_{30} \\ \phi_{40} \\ \phi_{50} \\ \phi_{60} \\ \phi_{70} \\ \phi_{80} \end{bmatrix}$$

節点番号

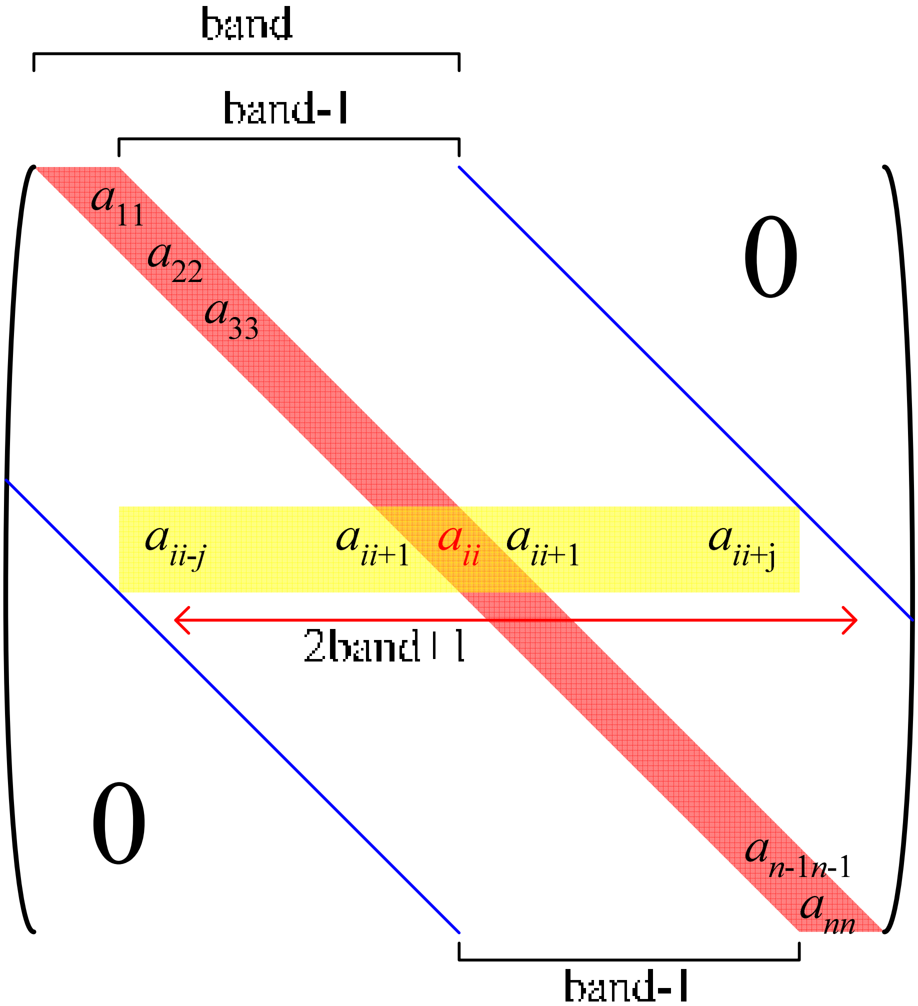
バンド幅とは行ごとの計算する必要のある変数の幅です。これは、節点番号の配置で異ります。



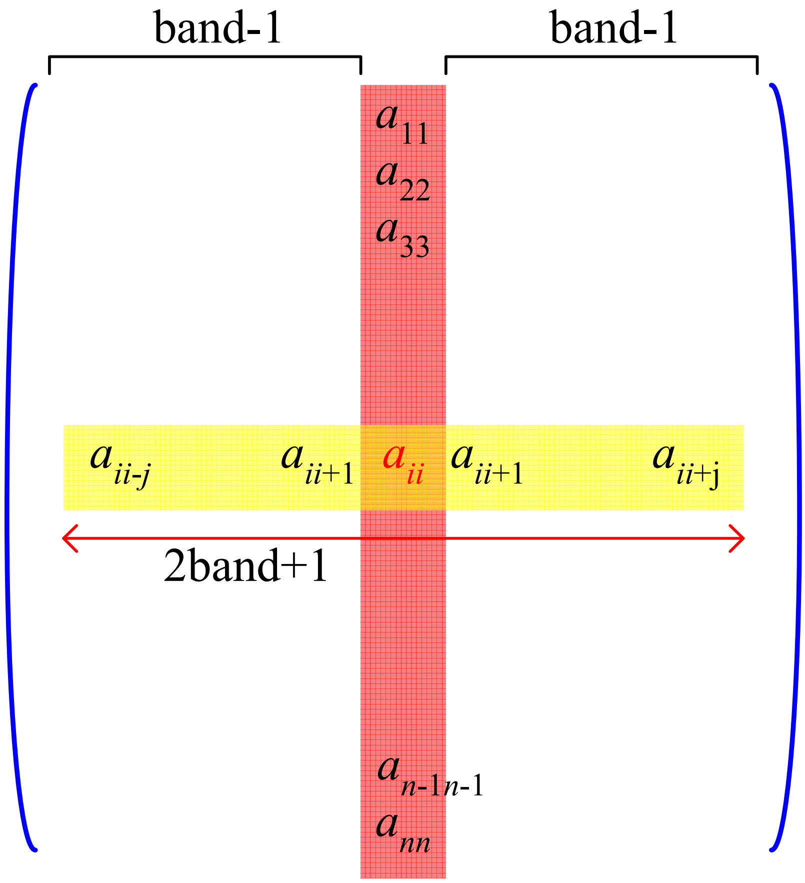
最適化前　　最適化後

上図の様にできるだけ近い節点には近い番号を割り振るとバンド幅が小さくなります。

バンド幅の詳細は下図の様になります。

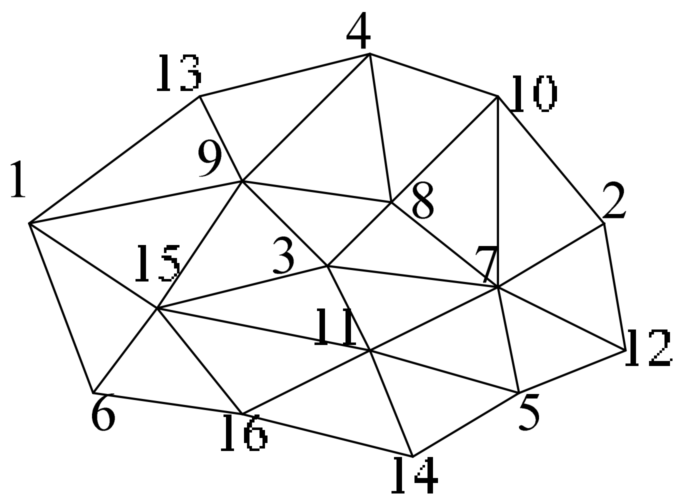


バンド幅が小さくなれば、下図の様により小さな行列式を解けばよいことになります。



バンド幅最適化の方法

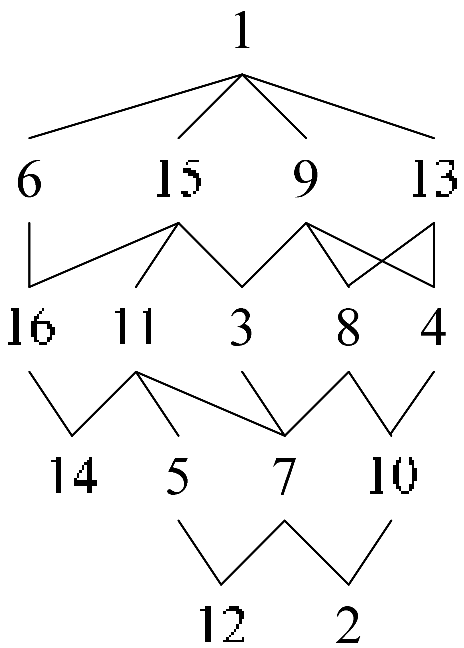
バンド幅の最適化とは、隣接する節点間の節点番号の差を最小にすることを意味しています。ここでは、バンド幅が最小になる節点の割り振り方について考えてみます。



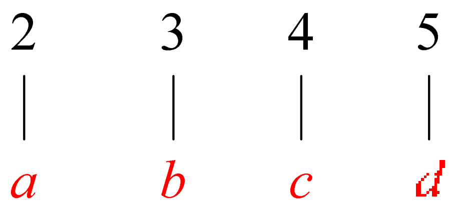
上図の領域について、バンド幅の最小化を考えてみます。この領域のバンド幅は、

(15 - 1) + 1 = 15

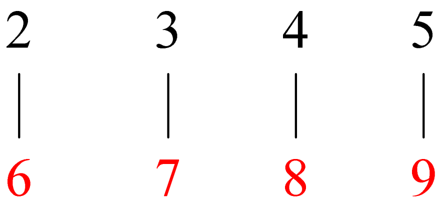
となります。バンド幅を最適化するために、節点を分木で表示すると下図となります。



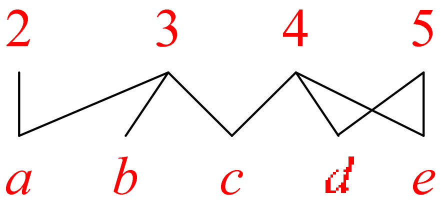
ここで、下図の様な場合の節点配置を考えてみます。



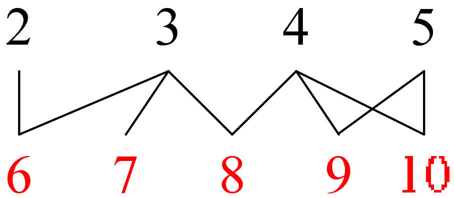
この場合は、より小さい節点番号に小さい順に節点を割り振れば、バンド幅は最小となります。



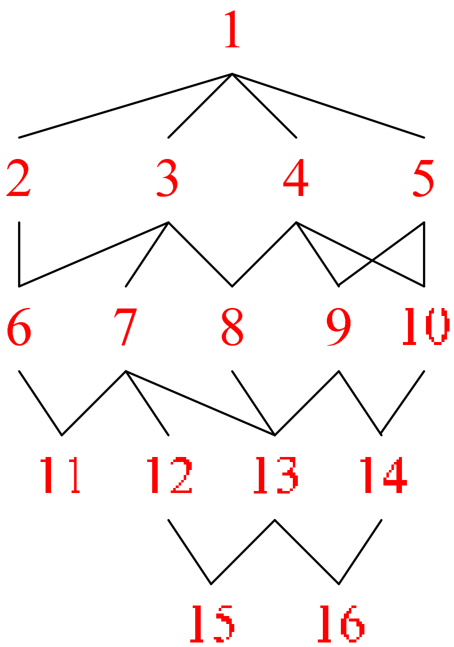
次に、枝が結合・分離した場合を考えて見ます。



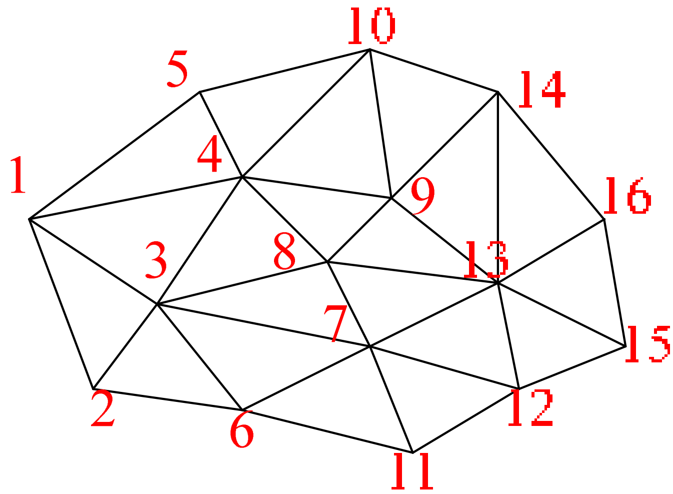
この場合も同様に節点番号の小さい順に節点を割り振れば、バンド幅は最小となります。



つまり、バンド幅を最小化するためには、節点番号のより小さい節点に、節点番号のより小さい節点から順番に割り振ればよいことになります。従って、分木で表した場合、ある段について左の枝から節点番号の小さい順に節点を割り振った場合、次の段についても左の枝から小さい順に節点を割り振ればバンド幅は最小になります。赤字が新しい節点番号です。



従って、新しい節点番号の領域は下図となります。



この図のバンド幅は、

$(10 - 4) + 1 = 7$

となります。