使用する単位

　伝熱工学では次の単位を使用します。

単位	記号	説明
温度	[K]	寒暖の度合いを数量で表したもの
熱量	[J]	熱エネルギーの大きさを表したもの
質量	[kg]	物質の重さ
距離、長さ	[m]	物質間の距離。物質の長さ
時間	[s]	

使用する物性値

比熱 C_p, C_v[J/kg・K]

1kg の物質の温度を 1 度上げるのに必要な熱量を表します。C_p[J/kg・K]は定圧比熱と呼ばれ、圧力一定の条件下での比熱を表します。一方、C_V[J/kg・K]は定積比熱と呼ばれ、体積一定の条件下での比熱です。

比熱が大きい : なかなか温度が上がらない。温度を上げるために多くの熱量を必要とする。例 : 水

比熱が小さい : 温度が上がりやすい。少量の熱で温度が上昇する。例 : 空気

従って、比熱の大きな物質の僅かな温度差で、比熱の小さな物質の温度を大きく上昇させることも可能になります。

熱伝導率 k[J/m・K・s]

熱の伝わりやすさを表します。物質の保温性を表しているとも言えます。

熱伝導率が大きい : 熱が速く伝わる。温まりやすく、冷めやすい。例 : 銅

熱伝導率が小さい : 熱がなかなか伝わらない。一旦温まると、なかなか冷めない。例 : 発泡スチロール

熱伝導率が小さい物質は、保温材・蓄熱媒体として用いられます。蓄熱媒体の間に炭素などの熱伝導率の高い物質を挟んで効率を上げるといった使用方法もあるようです。

境膜伝熱係数 h[J/m^2・K・s]

気体・液体中の熱の伝わりやすさを表しています。それらの流体がどのような状態か（層流なのか乱流なのか等）で係数の大きさも変わってきます。

熱流束

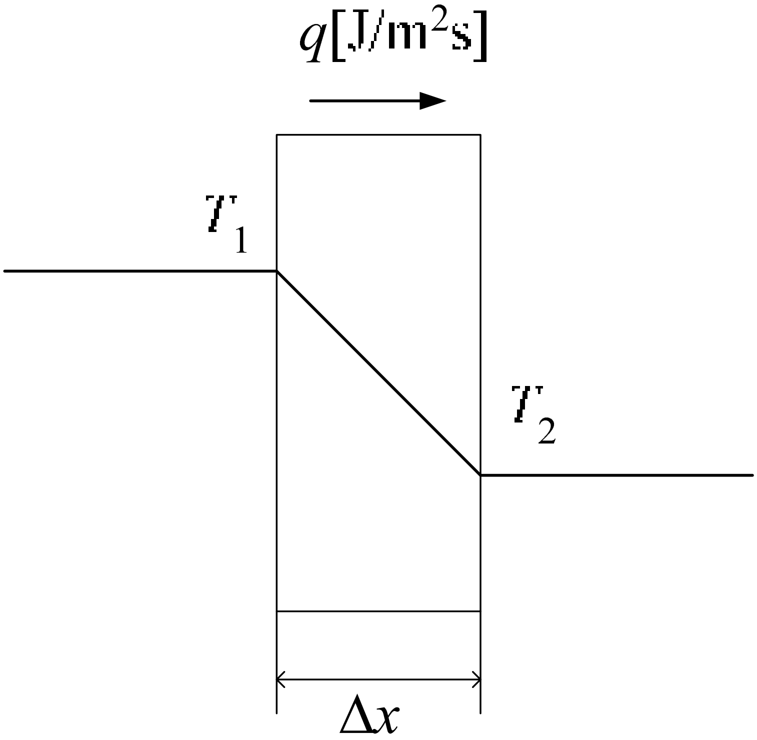
伝熱工学では、熱流束 q[J/m^2・s]という物理量が使用されます。これは、単位面積・単位時間当たりの熱量を表しています。簡易的に言えば、熱が伝わる勢いを表しています。流束とは、「単位面積・単位時間当たりの」を意味しています。伝導伝熱による熱流束は、熱伝導度 k[J/m・K・s]を用いて次式で定義されます。

$$q = -k\frac{dT}{dx}$$

これをフーリエの法則と言います。

伝熱の種類

伝導伝熱



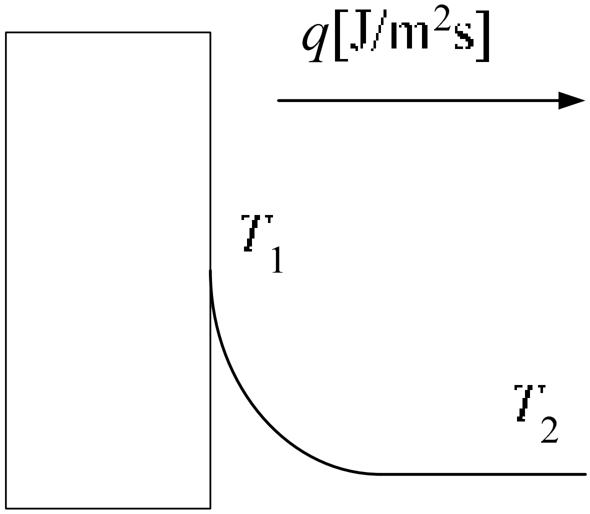
伝導伝熱とは、流れによらず熱が伝わる現象です。固体中等の流れの無い場所でも熱が伝わる現象は、伝導伝熱によるものです。伝導伝熱による熱流束 q[J/m^2・s]は次式で定義されます。

$$q = -k \lim_{\Delta x \to 0} \frac{T_2 - T_1}{\Delta x}$$

$$= -k \frac{dT}{dx}$$

伝熱の種類

対流伝熱

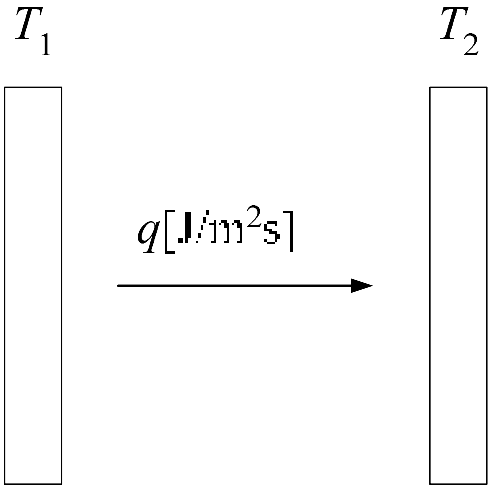


対流伝熱とは、流れ（対流）によって熱が伝わる現象です。流れの状態は、境膜伝熱係数 h[J/m^2・K・s]によって表されます。対流伝熱による熱流束 q[J/m^2・s]は次式で表されます。

$$q = -h(T_2 - T_1)$$

伝熱の種類

放射伝熱



放射伝熱による熱流束 q[J/m^2・s]は次式で表されます。

$$q = -\sigma(T_2^4 - T_1^4)$$

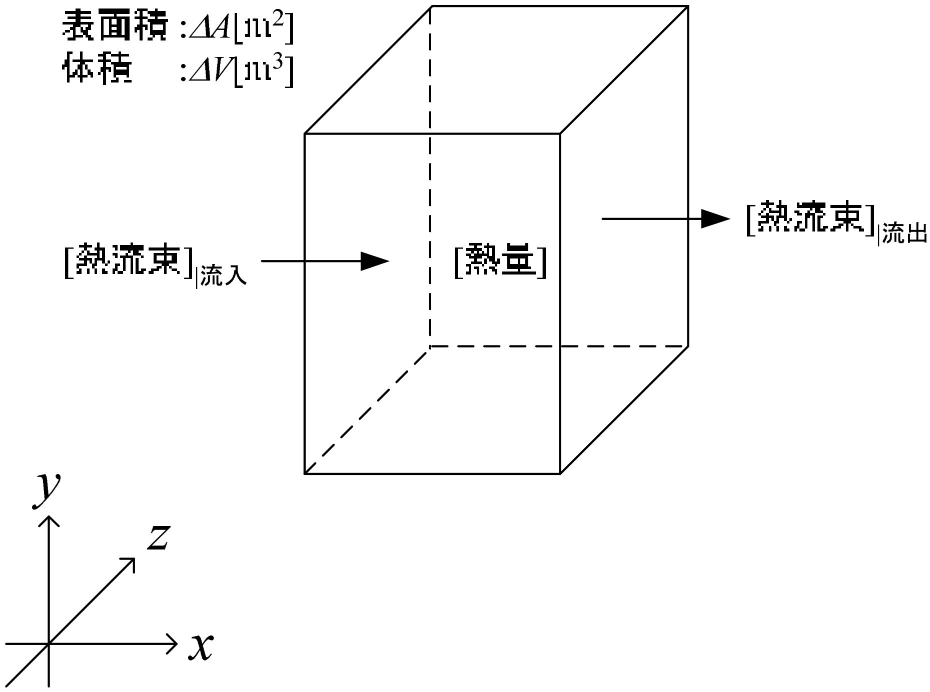
ここで、σ[J/m^2・K^4・s]は、ステファン－ボルツマン係数で

$$\sigma = 5.669\times10^{-8}$$

です。

熱収支式

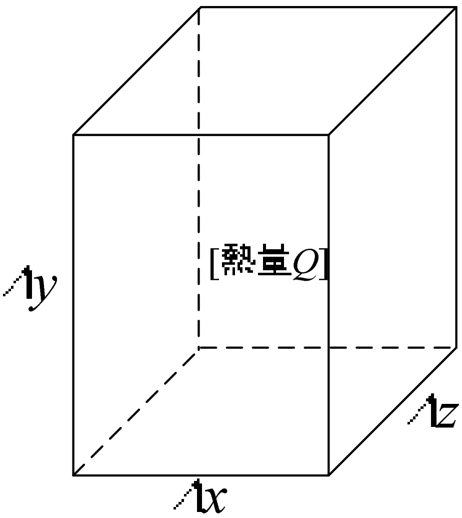
熱収支式は、下図のような微小領域内の熱収支を取ることで導出されます。



[蓄積される熱量]×ΔV＝[流入する熱流束]×ΔA -[流出する熱流束]×ΔA

熱収支式

蓄積量



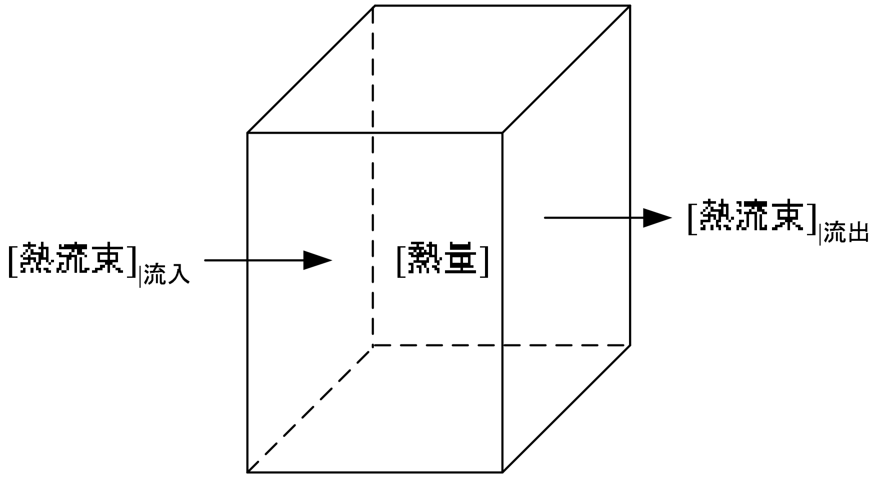
微小区間内に蓄積される熱流量 Q[J/s]は次式で表されます。

$$Q = \rho C_p \frac{T_{t+\Delta t} - T_t}{\Delta t} \Delta x \Delta y \Delta z$$

これは、微小時間内に蓄積される熱量を表しています。

熱収支式

対流伝熱



対流によって流入・流出する熱流量 $Q_{\mathrm{in}}, Q_{\mathrm{out}}$[J/s]は、次式で表されます。

流入

$$\begin{aligned} Q_{in} &= C_p(F_x T_{|x} + F_y T_{|y} + F_z T_{|z}) \\ &= C_p(\rho \Delta y \Delta z v_x T_{|x} + \rho \Delta z \Delta x v_y T_{|y} + \rho \Delta x \Delta y v_z T_{|z}) \\ &= \rho C_p(\Delta y \Delta z v_x T_{|x} + \Delta z \Delta x v_y T_{|y} + \Delta x \Delta y v_z T_{|z}) \end{aligned}$$

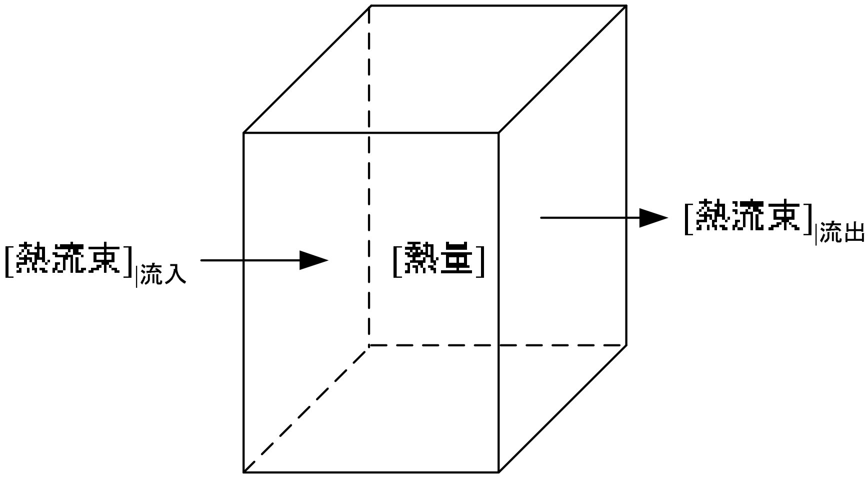
流出

$$\begin{aligned} Q_{out} &= -C_p(F_x T_{|x+\Delta x} + F_y T_{|y+\Delta y} + F_z T_{|z+\Delta z}) \\ &= -C_p(\rho \Delta y \Delta z v_x T_{|x+\Delta x} + \rho \Delta z \Delta x v_y T_{|y+\Delta y} + \rho \Delta x \Delta y v_z T_{|z+\Delta z}) \\ &= -\rho C_p(\Delta y \Delta z v_x T_{|x+\Delta x} + \Delta z \Delta x v_y T_{|y+\Delta y} + \Delta x \Delta y v_z T_{|z+\Delta z}) \end{aligned}$$

ここで、F_x, F_y, F_z[kg/s]は、それぞれ x, y, z 軸方向の質量流量を表しています。

熱収支式

伝導伝熱



伝導伝熱によって流入・流出する熱流量 Q_{in}, Q_{out}[J/s]は、次式で表されます。

流入

$$Q_{in} = \Delta y \Delta z q_x + \Delta z \Delta x q_y + \Delta x \Delta y q_z$$

$$= -\Delta y \Delta z k \frac{dT}{dx}_{|x} - \Delta z \Delta x k \frac{dT}{dy}_{|y} - \Delta x \Delta y k \frac{dT}{dz}_{|z}$$

流出

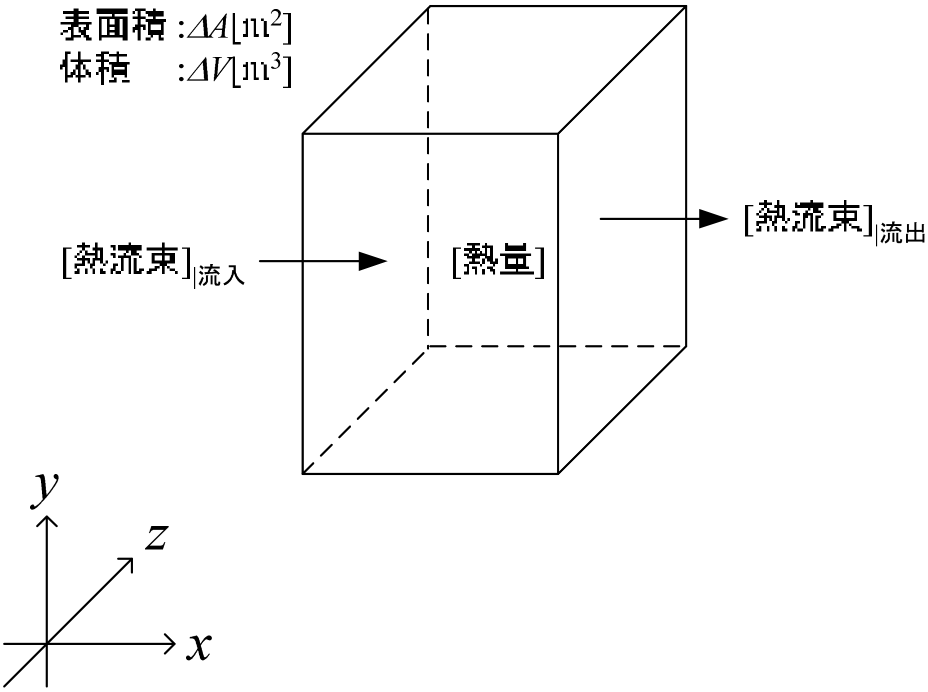
$$Q_{out} = -\Delta y \Delta z q_{x+\Delta x} - \Delta z \Delta x q_{y+\Delta y} - \Delta x \Delta y q_{z+\Delta z}$$

$$= \Delta y \Delta z k \frac{dT}{dx}_{|x+\Delta x} + \Delta z \Delta x k \frac{dT}{dy}_{|y+\Delta y} + \Delta x \Delta y k \frac{dT}{dz}_{|z+\Delta z}$$

ここで、k[J/m・K・s]は、熱伝導率を表しています。

熱収支式

収支式の導出



流入・流出する熱流量の収支を取ると、収支式は、次式で表されます。

$$
\begin{aligned}
\rho C_p \frac{T_{t+\Delta t} - T_t}{\Delta t} \Delta x \Delta y \Delta z = {} & \rho C_p \left(\Delta y \Delta z v_x T_{|x} + \Delta z \Delta x v_y T_{|y} + \Delta x \Delta y v_z T_{|z} \right) \\
& - \rho C_p \left(\Delta y \Delta z v_x T_{|x+\Delta x} + \Delta z \Delta x v_y T_{|y+\Delta y} + \Delta x \Delta y v_z T_{|z+\Delta z} \right) \\
& - \Delta y \Delta z k \frac{dT}{dx}_{|x} - \Delta z \Delta x k \frac{dT}{dy}_{|y} - \Delta x \Delta y k \frac{dT}{dz}_{|z} \\
& + \Delta y \Delta z k \frac{dT}{dx}_{|x+\Delta x} + \Delta z \Delta x k \frac{dT}{dy}_{|y+\Delta y} + \Delta x \Delta y k \frac{dT}{dz}_{|z+\Delta z}
\end{aligned}
$$

これは、x 軸方向の収支を表してます。

両辺を$\Delta x\,\Delta y\,\Delta z$で割ると

$$\rho C_p \frac{T_{t+\Delta t}-T_t}{\Delta t}=\rho C_p(\frac{v_{x|x}T_{|x}}{\Delta x}+\frac{v_{y|y}T_{|x}}{\Delta y}+\frac{v_{z|z}T_{|x}}{\Delta z})$$

$$-\rho C_p(\frac{v_{x|x+\Delta x}T_{|x+\Delta x}}{\Delta x}+\frac{v_{y|y+\Delta y}T_{|x+\Delta x}}{\Delta y}+\frac{v_{z|z+\Delta z}T_{|x+\Delta x}}{\Delta z})$$

$$-k\frac{dT}{dx}_{|x}/\Delta x-k\frac{dT}{dx}_{|y}/\Delta y-k\frac{dT}{dx}_{|z}/\Delta z$$

$$+k\frac{dT}{dx}_{|x+\Delta x}/\Delta x+k\frac{dT}{dx}_{|y+\Delta y}/\Delta y+k\frac{dT}{dx}_{|z+\Delta z}/\Delta z$$

整理して

$$\rho C_p \frac{T_{t+\Delta t}-T_t}{\Delta t}$$

$$+\rho C_p(\frac{v_{x|x+\Delta x}T_{|x+\Delta x}-v_{x|x}T_{|x}}{\Delta x})$$

$$+\rho C_p(\frac{v_{y|y+\Delta y}T_{|y+\Delta y}-v_{y|y}T_{|y}}{\Delta y})$$

$$+\rho C_p(\frac{v_{z|z+\Delta z}T_{|x+\Delta x}-v_{z|z}T_{|z}}{\Delta z})$$

$$=k(\frac{\frac{dT}{dx}_{|x+\Delta x}-\frac{dT}{dx}_{|x}}{\Delta x}+\frac{\frac{dT}{dy}_{|y+\Delta y}-\frac{dT}{dy}_{|y}}{\Delta y}+\frac{\frac{dT}{dz}_{|z+\Delta z}-\frac{dT}{dz}_{|z}}{\Delta z})$$

$\Delta x\to 0$とすると

$$\rho C_p\{\frac{dT}{dt}+\frac{d(v_xT)}{dx}+\frac{d(v_yT)}{dy}+\frac{d(v_zT)}{dz}\}=k(\frac{d^2T}{d^2x}+\frac{d^2T}{d^2y}+\frac{d^2T}{d^2z})$$

従って、

$$\rho C_p(\frac{dT}{dt}+v_x\frac{dT}{dx}+v_y\frac{dT}{dy}+v_z\frac{dT}{dz})=k(\frac{d^2T}{d^2x}+\frac{d^2T}{d^2y}+\frac{d^2T}{d^2z})$$

ベクトル表示すると

$$\rho C_p \frac{DT}{Dt} = k\Delta T$$

上式は、熱伝導率が一定の場合です。

・無次元化

無次元化とは、収支式に使用されている変数の次元を無次元[-]にする操作です。**無次元化することで収支式の定数は無次元数のみとなりなす。**逆に言えば、ある無次元数で計算された結果は、その無次元数の変数をどのようにとっても変わりません。速度や長さ、密度などが異なっても無次元数が等しければ同じ現象が現れます。変数は次式で無次元化されます。ここで、x_0[m]は代表長さ、v_0[m/s]は代表速度です。

座標の無次元化

$$X=\frac{x}{x_0}, Y=\frac{y}{x_0}, Z=\frac{z}{x_0}$$

速度の無次元化

$$V_x=\frac{v_x}{v_0}, V_y=\frac{v_y}{v_0}, V_z=\frac{v_z}{v_0}$$

温度の無次元化

$$\Theta=\frac{T-T_0}{T_1-T_0}$$

時間の無次元化

$$\tau=\frac{v_0}{x_0}t$$

・熱収支式の無次元化

無次元化前

$$\rho C_p\frac{DT}{Dt}=-(\nabla\cdot\vec{q})$$

ここで、$\vec{q}$ [J/m^{2}s]は熱流束で、

$$\vec{q}=-k(\vec{\delta}_x\frac{\partial T}{\partial x}+\vec{\delta}_y\frac{\partial T}{\partial y}+\vec{\delta}_z\frac{\partial T}{\partial z})$$

成分表示すると

$$\rho C_p(\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z})=k(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2})$$

無次元化後

$$\frac{D\Theta}{D\tau}=-(\nabla\cdot\vec{q}^*)$$

ここで、$\vec{q}^*$ [-]は熱流束で、

$$\vec{q}^* = -\frac{1}{Pe}(\vec{\delta}_x \frac{\partial T}{\partial x} + \vec{\delta}_y \frac{\partial T}{\partial y} + \vec{\delta}_z \frac{\partial T}{\partial z})$$

成分表示すると

$$\frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + V_z \frac{\partial \Theta}{\partial Z} = \frac{1}{Pe}(\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial^2 \Theta}{\partial Z^2})$$

また、

$$Pe = \frac{\rho C_p v_0 x_0}{k}$$

ペクレ数 Pe は、対流伝熱と伝導伝熱の比を表しています。

熱収支式の離散化

ここでは、次の要素についての離散化を紹介します。

- 3 角形 1 次要素
- 4 面体 1 次要素

熱収支式の離散化

3 角形 1 次要素

$$\phi = \frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y} = 0$$

要素に 3 角形を用いた場合、

$$\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \phi\, dS = \int_S \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} dS = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x,1} + N_2 V_{x,2} + N_3 V_{x,3} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} V_{x,1} \\ V_{x,2} \\ V_{x,3} \end{Bmatrix} = [N]\{V_x\} = 0$$

$$V_y = N_1 V_{y,1} + N_2 V_{y,2} + N_3 V_{y,3} = 0$$

$$\Theta = N_1 \Theta_1 + N_2 \Theta_2 + N_3 \Theta_3$$

となるので、

$$\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \left(\frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y} \right) dS$$

$$= \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial \tau} dS + V_x \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial X} dS + V_y \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial Y} dS$$

$$+ \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial q_x^*}{\partial X} dS + \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial q_y^*}{\partial Y} dS$$

展開すると

$$\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \left(\frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y}\right) dS$$

$$= \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial \tau} dS + V_x \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial X} dS + V_y \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial \Theta}{\partial Y} dS$$

$$+ \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial q_x^*}{\partial X} dS + \int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \frac{\partial q_y^*}{\partial Y} dS$$

$$= \int_S [N]^T \frac{\partial \Theta}{\partial \tau} dS + V_x \int_S [N]^T \frac{\partial \Theta}{\partial X} dS + V_y \int_S [N]^T \frac{\partial \Theta}{\partial Y} dS$$

$$+ \int_S [N]^T \frac{\partial q_x^*}{\partial X} dS + \int_S [N]^T \frac{\partial q_y^*}{\partial Y} dS$$

グリーン・ガウスの定理より

$$= \int_S [N]^{\mathrm{T}} \frac{[N](\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau})}{\Delta \tau} dS$$

$$+ V_x \int_S [N]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial X} dS + V_y \int_S [N]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Y} dS$$

$$+ \int_L [N]^{\mathrm{T}} q_x^* n_x dL - \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} q_x^* dS$$

$$+ \int_L [N]^{\mathrm{T}} q_y^* n_y dL - \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} q_y^* dS$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}} [N] dS \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\} \\
&- \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} q_x^* dS - \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} q_y^* dS \\
&+ \int_L [N]^{\mathrm{T}} (q_x^* n_x + q_y^* n_y) dL
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}} [N] dS \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\} \\
&- \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left(-\frac{1}{Pe}\frac{\partial \Theta}{\partial X}\right) dS \\
&- \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left(-\frac{1}{Pe}\frac{\partial \Theta}{\partial Y}\right) dS \\
&+ Nu(\Theta - \Theta_h) \int_L [N]^{\mathrm{T}} dL
\end{aligned}
$$

$$
\begin{aligned}
&= \int_S [N]^{\mathrm{T}} [N] dS \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\} \\
&+ \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial \Theta}{\partial X} dS + \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial \Theta}{\partial Y} dS \\
&+ Nu(\Theta - \Theta_h) \int_L [N]^{\mathrm{T}} dL
\end{aligned}
$$

$$= \int_S [N]^{\mathrm{T}} [N] dS \, \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\}$$

$$+ \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial X} dS + \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Y} dS$$

$$+ Nu(\Theta - \Theta_h) \int_L [N]^{\mathrm{T}} dL$$

$$= \int_S [N]^{\mathrm{T}} [N] dS \, \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + V_y \int_S [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\}$$

$$+ \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dS \{\Theta\} + \frac{1}{Pe} \int_S \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dS \{\Theta\}$$

$$+ Nu(\Theta - \Theta_h) \int_L [N]^{\mathrm{T}} dL$$

$$= \int_S \begin{bmatrix} N_1 N_1 & N_1 N_2 & N_1 N_3 \\ N_2 N_1 & N_2 N_2 & N_2 N_3 \\ N_3 N_1 & N_3 N_2 & N_3 N_3 \end{bmatrix} dS \, \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+ V_x \int_S \begin{bmatrix} N_1 \frac{\partial N_1}{\partial X} & N_1 \frac{\partial N_2}{\partial X} & N_1 \frac{\partial N_3}{\partial X} \\ N_2 \frac{\partial N_1}{\partial X} & N_2 \frac{\partial N_2}{\partial X} & N_2 \frac{\partial N_3}{\partial X} \\ N_3 \frac{\partial N_1}{\partial X} & N_3 \frac{\partial N_2}{\partial X} & N_3 \frac{\partial N_3}{\partial X} \end{bmatrix} dS \{\Theta\}$$

$$+ V_y \int_S \begin{bmatrix} N_1 \frac{\partial N_1}{\partial Y} & N_1 \frac{\partial N_2}{\partial Y} & N_1 \frac{\partial N_3}{\partial Y} \\ N_2 \frac{\partial N_1}{\partial Y} & N_2 \frac{\partial N_2}{\partial Y} & N_2 \frac{\partial N_3}{\partial Y} \\ N_3 \frac{\partial N_1}{\partial Y} & N_3 \frac{\partial N_2}{\partial Y} & N_3 \frac{\partial N_3}{\partial Y} \end{bmatrix} dS \{\Theta\}$$

$$+\frac{1}{Pe}\int_S \begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X} \end{bmatrix} dS\{\Theta\}$$

$$+\frac{1}{Pe}\int_S \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial Y} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial Y} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial Y} \end{bmatrix} dS\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)\int_L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dL$$

$$=\int_S \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 \\ L_2L_1 & L_2L_2 & L_2L_3 \\ L_3L_1 & L_3L_2 & L_3L_3 \end{bmatrix} dS\,\frac{\{\Theta\}^{\tau+\Delta\tau}-\{\Theta\}^{\tau}}{\Delta\tau}$$

$$+V_x\int_S \frac{1}{2A}\begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} \end{bmatrix} dS\{\Theta\}$$

$$+V_y\int_S \frac{1}{2A}\begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} \end{bmatrix} dS\{\Theta\}$$

$$+\frac{1}{Pe}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix} dS\{\Theta\}$$

$$+\frac{1}{Pe}\int_S \frac{1}{4A^2}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix} dS\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)\int_L \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} dL$$

ここで、

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!}2A$$

$$\int_S L_i L_j dS = \begin{cases} \dfrac{1!1!}{(1+1+2)!}2A = \dfrac{2}{4!}A = \dfrac{1}{12}A & (i \neq j) \\ \dfrac{2!}{(1+1+2)!}2A = \dfrac{4}{4!}V = \dfrac{1}{6}A & (i = j) \end{cases}$$

$$\int_S L_i dS = \frac{1!}{(1+2)!}2A = \frac{2}{3!}A = \frac{1}{3}A$$

$$\int_L L_1^p L_2^q dL = \frac{p!q!}{(p+q)!}L$$

$$\int_L L_i dL = \frac{1}{1!}L = L$$

$$= \frac{1}{12}A \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+V_x \frac{1}{2A}\frac{A}{3} \begin{bmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \end{bmatrix} \{\Theta\}$$

$$+V_y \frac{1}{2A}\frac{A}{3} \begin{bmatrix} c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \end{bmatrix} \{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{4A^2}A\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix}\{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{4A^2}A\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix}\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$=\frac{1}{12}A\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\frac{\{\Theta\}^{\tau+\Delta\tau}-\{\Theta\}^{\tau}}{\Delta\tau}$$

$$+V_x\frac{1}{6}\begin{bmatrix} c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \\ c_{1x} & c_{2x} & c_{3x} \end{bmatrix}\{\Theta\}$$

$$+V_y\frac{1}{6}\begin{bmatrix} c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \\ c_{1y} & c_{2y} & c_{3y} \end{bmatrix}\{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{4A}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} \end{bmatrix}\{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{4A}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} \end{bmatrix}\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)n_iL\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$
\begin{aligned}
&= [C]\frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} + V_x[C_x]\{\Theta\} + V_y[C_y]\{\Theta\} \\
&+ \frac{1}{Pe}[S_{xx}]\{\Theta\} + \frac{1}{Pe}[S_{yy}]\{\Theta\} \\
&+ Nu(\Theta - \Theta_h)L\begin{bmatrix}1\\1\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= [C]\frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} + (V_x[C_x] + V_y[C_y])\{\Theta\} \\
&+ \frac{1}{Pe}([S_{xx}] + [S_{yy}])\{\Theta\} \\
&+ Nu(\Theta - \Theta_h)L\begin{bmatrix}1\\1\end{bmatrix} \\
&= \begin{bmatrix}0\\0\\0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&[C]\frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} + (V_x[C_x] + V_y[C_y])\{\Theta\} \\
&+ \frac{1}{Pe}([S_{xx}] + [S_{yy}])\{\Theta\} \\
&+ Nu(\Theta - \Theta_h)L\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}
\end{aligned}
$$

陰解法

$$
\begin{aligned}
&\frac{[C]}{\Delta\tau}\{\Theta\}^{\tau+\Delta\tau} + (V_x[C_x] + V_y[C_y])\{\Theta\}^{\tau+\Delta\tau} \\
&+ \frac{1}{Pe}([S_{xx}] + [S_{yy}])\{\Theta\}^{\tau+\Delta\tau} \\
&= \frac{[C]}{\Delta\tau}\{\Theta\}^{\tau} - Nu(\Theta - \Theta_h)L\begin{bmatrix}1\\1\end{bmatrix}
\end{aligned}
$$

熱収支式の離散化
　4 面体 1 次要素

$$\phi = \frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + V_z \frac{\partial \Theta}{\partial Z} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y} + \frac{\partial q_z^*}{\partial Z} = 0$$

要素に 4 面体を用いた場合、

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \phi \, dV = \int_V \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} dV = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$V_x = N_1 V_{x,1} + N_2 V_{x,2} + N_3 V_{x,3} + N_4 V_{x,4} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} V_{x,1} \\ V_{x,2} \\ V_{x,3} \\ V_{x,4} \end{Bmatrix} = [N]\{V_x\} = 0$$

$$V_y = N_1 V_{y,1} + N_2 V_{y,2} + N_3 V_{y,3} + N_4 V_{y,4} = 0$$

$$V_z = N_1 V_{z,1} + N_2 V_{z,2} + N_3 V_{z,3} + N_4 V_{z,4} = 0$$

$$\Theta = N_1 \Theta_1 + N_2 \Theta_2 + N_3 \Theta_3 + N_4 \Theta_4$$

となるので、

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left(\frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + V_z \frac{\partial \Theta}{\partial Z} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y} + \frac{\partial q_z^*}{\partial Z} \right) dV$$

$$= \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial \tau} dV + V_x \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial X} dV + V_y \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial Y} dV + V_z \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial Z} dV$$

$$+ \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_x^*}{\partial X} dV + \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_y^*}{\partial Y} dV + \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_z^*}{\partial Z} dV$$

展開すると

$$\int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \left(\frac{\partial \Theta}{\partial \tau} + V_x \frac{\partial \Theta}{\partial X} + V_y \frac{\partial \Theta}{\partial Y} + V_z \frac{\partial \Theta}{\partial Z} + \frac{\partial q_x^*}{\partial X} + \frac{\partial q_y^*}{\partial Y} + \frac{\partial q_z^*}{\partial Z}\right) dV$$

$$= \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial \tau} dV + V_x \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial X} dV + V_y \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial Y} dV + V_z \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial \Theta}{\partial Z} dV$$

$$+ \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_x^*}{\partial X} dV + \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_y^*}{\partial Y} dV + \int_V \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \frac{\partial q_z^*}{\partial Z} dV$$

$$= \int_V [N]^T \frac{\partial \Theta}{\partial \tau} dV + V_x \int_V [N]^T \frac{\partial \Theta}{\partial X} dV + V_y \int_V [N]^T \frac{\partial \Theta}{\partial Y} dV + V_z \int_V [N]^T \frac{\partial \Theta}{\partial Z} dV$$

$$+ \int_V [N]^T \frac{\partial q_x^*}{\partial X} dV + \int_V [N]^T \frac{\partial q_y^*}{\partial Y} dV + \int_V [N]^T \frac{\partial q_z^*}{\partial Z} dV$$

グリーン・ガウスの定理より

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}} \frac{[N](\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau})}{\Delta\tau} dV \\
&+ V_x \int_V [N]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial X} dV + V_y \int_V [N]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Y} dV + V_z \int_V [N]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Z} dV \\
&+ \int_S [N]^{\mathrm{T}} q_x^* n_x dS - \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} q_x^* dV \\
&+ \int_S [N]^{\mathrm{T}} q_y^* n_y dS - \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} q_y^* dV \\
&+ \int_S [N]^{\mathrm{T}} q_z^* n_z dS - \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} q_z^* dV
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + V_z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&- \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} q_x^* dV - \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} q_y^* dV - \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} q_z^* dV \\
&+ \int_S [N]^{\mathrm{T}} (q_x^* n_x + q_y^* n_y + q_z^* n_z) dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}} [N] dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + V_z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&- \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left(-\frac{1}{Pe} \frac{\partial \Theta}{\partial X}\right) dV \\
&- \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left(-\frac{1}{Pe} \frac{\partial \Theta}{\partial Y}\right) dV \\
&- \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \left(-\frac{1}{Pe} \frac{\partial \Theta}{\partial Z}\right) dV \\
&+ Nu(\Theta - \Theta_h) \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}}[N] dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + V_z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&+ \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial \Theta}{\partial X} dV + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial \Theta}{\partial Y} dV + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial \Theta}{\partial Z} dV \\
&+ Nu(\Theta - \Theta_h) \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}}[N] dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + V_z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&+ \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial X} dV + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Y} dV + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \frac{\partial [N]\{\Theta\}}{\partial Z} dV \\
&+ Nu(\Theta - \Theta_h) \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
\begin{aligned}
&= \int_V [N]^{\mathrm{T}}[N] dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau} \\
&+ V_x \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + V_y \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + V_z \int_V [N]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&+ \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial X}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial X}\right] dV \{\Theta\} + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Y}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Y}\right] dV \{\Theta\} + \frac{1}{Pe} \int_V \left[\frac{\partial N}{\partial Z}\right]^{\mathrm{T}} \left[\frac{\partial N}{\partial Z}\right] dV \{\Theta\} \\
&+ Nu(\Theta - \Theta_h) \int_S [N]^{\mathrm{T}} dS
\end{aligned}
$$

$$
= \int_V \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 & N_1N_4 \\ N_2N_1 & N_2N_2 & N_2N_3 & N_2N_4 \\ N_3N_1 & N_3N_2 & N_3N_3 & N_3N_4 \\ N_4N_1 & N_4N_2 & N_4N_3 & N_4N_4 \end{bmatrix} dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}
$$

$$+V_x\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial X} & N_1\frac{\partial N_2}{\partial X} & N_1\frac{\partial N_3}{\partial X} & N_1\frac{\partial N_4}{\partial X} \\ N_2\frac{\partial N_1}{\partial X} & N_2\frac{\partial N_2}{\partial X} & N_2\frac{\partial N_3}{\partial X} & N_2\frac{\partial N_4}{\partial X} \\ N_3\frac{\partial N_1}{\partial X} & N_3\frac{\partial N_2}{\partial X} & N_3\frac{\partial N_3}{\partial X} & N_3\frac{\partial N_4}{\partial X} \\ N_4\frac{\partial N_1}{\partial X} & N_4\frac{\partial N_2}{\partial X} & N_4\frac{\partial N_3}{\partial X} & N_4\frac{\partial N_4}{\partial X} \end{bmatrix} dV\{\Theta\}$$

$$+V_y\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Y} & N_1\frac{\partial N_2}{\partial Y} & N_1\frac{\partial N_3}{\partial Y} & N_1\frac{\partial N_4}{\partial Y} \\ N_2\frac{\partial N_1}{\partial Y} & N_2\frac{\partial N_2}{\partial Y} & N_2\frac{\partial N_3}{\partial Y} & N_2\frac{\partial N_4}{\partial Y} \\ N_3\frac{\partial N_1}{\partial Y} & N_3\frac{\partial N_2}{\partial Y} & N_3\frac{\partial N_3}{\partial Y} & N_3\frac{\partial N_4}{\partial Y} \\ N_4\frac{\partial N_1}{\partial Y} & N_4\frac{\partial N_2}{\partial Z} & N_4\frac{\partial N_3}{\partial Y} & N_4\frac{\partial N_4}{\partial Y} \end{bmatrix} dV\{\Theta\}$$

$$+V_z\int_V \begin{bmatrix} N_1\frac{\partial N_1}{\partial Z} & N_1\frac{\partial N_2}{\partial Z} & N_1\frac{\partial N_3}{\partial Z} & N_1\frac{\partial N_4}{\partial Z} \\ N_2\frac{\partial N_1}{\partial Z} & N_2\frac{\partial N_2}{\partial Z} & N_2\frac{\partial N_3}{\partial Z} & N_2\frac{\partial N_4}{\partial Z} \\ N_3\frac{\partial N_1}{\partial Z} & N_3\frac{\partial N_2}{\partial Z} & N_3\frac{\partial N_3}{\partial Z} & N_3\frac{\partial N_4}{\partial Z} \\ N_4\frac{\partial N_1}{\partial Z} & N_4\frac{\partial N_2}{\partial Z} & N_4\frac{\partial N_3}{\partial Z} & N_4\frac{\partial N_4}{\partial Z} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_1}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_2}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_3}{\partial X}\frac{\partial N_1}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_2}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_3}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_4}{\partial X} \\ \frac{\partial N_1}{\partial X}\frac{\partial N_4}{\partial X} & \frac{\partial N_2}{\partial X}\frac{\partial N_4}{\partial X} & \frac{\partial N_3}{\partial X}\frac{\partial N_4}{\partial X} & \frac{\partial N_4}{\partial X}\frac{\partial N_4}{\partial X} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_1}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_2}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_2}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_3}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_3}{\partial Y}\frac{\partial N_4}{\partial Y} \\ \frac{\partial N_4}{\partial Y}\frac{\partial N_1}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_2}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_3}{\partial Y} & \frac{\partial N_4}{\partial Y}\frac{\partial N_4}{\partial Y} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \begin{bmatrix} \frac{\partial N_1}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_1}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_2}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_2}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_3}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_3}{\partial Z}\frac{\partial N_4}{\partial Z} \\ \frac{\partial N_4}{\partial Z}\frac{\partial N_1}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_2}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_3}{\partial Z} & \frac{\partial N_4}{\partial Z}\frac{\partial N_4}{\partial Z} \end{bmatrix} dV\{\Theta\}$$

$$+ Nu(\Theta - \Theta_h)\int_S \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} dS$$

$$= \int_V \begin{bmatrix} L_1L_1 & L_1L_2 & L_1L_3 & L_1L_4 \\ L_2L_1 & L_2L_2 & L_2L_3 & L_2L_4 \\ L_3L_1 & L_3L_2 & L_3L_3 & L_3L_4 \\ L_4L_1 & L_4L_2 & L_4L_3 & L_4L_4 \end{bmatrix} dV \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+V_x\int_V \frac{1}{6V}\begin{bmatrix} L_1c_{1x} & L_1c_{2x} & L_1c_{3x} & L_1c_{4x} \\ L_2c_{1x} & L_2c_{2x} & L_2c_{3x} & L_2c_{4x} \\ L_3c_{1x} & L_3c_{2x} & L_3c_{3x} & L_3c_{4x} \\ L_4c_{1x} & L_4c_{2x} & L_4c_{3x} & L_4c_{4x} \end{bmatrix} dV\{\Theta\}$$

$$+V_y\int_V \frac{1}{6V}\begin{bmatrix} L_1c_{1y} & L_1c_{2y} & L_1c_{3y} & L_1c_{4y} \\ L_2c_{1y} & L_2c_{2y} & L_2c_{3y} & L_2c_{4y} \\ L_3c_{1y} & L_3c_{2y} & L_3c_{3y} & L_3c_{4y} \\ L_4c_{1y} & L_4c_{2y} & L_4c_{3y} & L_4c_{4y} \end{bmatrix} dV\{\Theta\}$$

$$+V_z\int_V \frac{1}{6V}\begin{bmatrix} L_1c_{1z} & L_1c_{2z} & L_1c_{3z} & L_1c_{4z} \\ L_2c_{1z} & L_2c_{2z} & L_2c_{3z} & L_2c_{4z} \\ L_3c_{1z} & L_3c_{2z} & L_3c_{3z} & L_3c_{4z} \\ L_4c_{1z} & L_4c_{2z} & L_4c_{3z} & L_4c_{4z} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} & c_{1x}c_{4x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} & c_{2x}c_{4x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} & c_{3x}c_{4x} \\ c_{4x}c_{1x} & c_{4x}c_{2x} & c_{4x}c_{3x} & c_{4x}c_{4x} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} & c_{1y}c_{4y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} & c_{2y}c_{4y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} & c_{3y}c_{4y} \\ c_{4y}c_{1y} & c_{4y}c_{2y} & c_{4y}c_{3y} & c_{4y}c_{4y} \end{bmatrix} dV\{\Theta\}$$

$$+\frac{1}{Pe}\int_V \frac{1}{36V^2}\begin{bmatrix} c_{1z}c_{1z} & c_{1z}c_{2z} & c_{1z}c_{3z} & c_{1z}c_{4z} \\ c_{2z}c_{1z} & c_{2z}c_{2z} & c_{2z}c_{3z} & c_{2z}c_{4z} \\ c_{3z}c_{1z} & c_{3z}c_{2z} & c_{3z}c_{3z} & c_{3z}c_{4z} \\ c_{4z}c_{1z} & c_{4z}c_{2z} & c_{4z}c_{3z} & c_{4z}c_{4z} \end{bmatrix} dV\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)\int_S \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} dS$$

ここで、

$$\int_V L_1^p L_2^q L_3^r L_4^s dV = \frac{p!q!r!s!}{(p+q+r+s+3)!}6V$$

$$\int_V L_i L_j dV = \begin{cases} \dfrac{1!1!}{(1+1+3)!}6V = \dfrac{6}{5!}V = \dfrac{1}{20}V & (i \neq j) \\ \dfrac{2!}{(1+1+3)!}6V = \dfrac{12}{5!}V = \dfrac{1}{10}V & (i = j) \end{cases}$$

$$\int_V L_i dV = \frac{1!}{(1+3)!}6V = \frac{6}{4!}V = \frac{1}{4}V$$

$$\int_S L_1^p L_2^q L_3^r dS = \frac{p!q!r!}{(p+q+r+2)!}2S$$

$$\int_S L_i dS = \frac{1}{(1+2)!}2S = \frac{1}{3}S$$

$$= \frac{1}{20}V \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$

$$+ V_x \frac{1}{6V}\frac{V}{4} \begin{bmatrix} c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \\ c_{1x} & c_{2x} & c_{3x} & c_{4x} \end{bmatrix} \{\Theta\}$$

$$+ V_y \frac{1}{6V}\frac{V}{4} \begin{bmatrix} c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \\ c_{1y} & c_{2y} & c_{3y} & c_{4y} \end{bmatrix} \{\Theta\}$$

$$+ V_z \frac{1}{6V}\frac{V}{4} \begin{bmatrix} c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \\ c_{1z} & c_{2z} & c_{3z} & c_{4z} \end{bmatrix} \{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{36V^2}V\begin{bmatrix} c_{1x}c_{1x} & c_{1x}c_{2x} & c_{1x}c_{3x} & c_{1x}c_{4x} \\ c_{2x}c_{1x} & c_{2x}c_{2x} & c_{2x}c_{3x} & c_{2x}c_{4x} \\ c_{3x}c_{1x} & c_{3x}c_{2x} & c_{3x}c_{3x} & c_{3x}c_{4x} \\ c_{4x}c_{1x} & c_{4x}c_{2x} & c_{4x}c_{3x} & c_{4x}c_{4x} \end{bmatrix}\{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{36V^2}V\begin{bmatrix} c_{1y}c_{1y} & c_{1y}c_{2y} & c_{1y}c_{3y} & c_{1y}c_{4y} \\ c_{2y}c_{1y} & c_{2y}c_{2y} & c_{2y}c_{3y} & c_{2y}c_{4y} \\ c_{3y}c_{1y} & c_{3y}c_{2y} & c_{3y}c_{3y} & c_{3y}c_{4y} \\ c_{4y}c_{1y} & c_{4y}c_{2y} & c_{4y}c_{3y} & c_{4y}c_{4y} \end{bmatrix}\{\Theta\}$$

$$+\frac{1}{Pe}\frac{1}{36V^2}V\begin{bmatrix} c_{1z}c_{1z} & c_{1z}c_{2z} & c_{1z}c_{3z} & c_{1z}c_{4z} \\ c_{2z}c_{1z} & c_{2z}c_{2z} & c_{2z}c_{3z} & c_{2z}c_{4z} \\ c_{3z}c_{1z} & c_{3z}c_{2z} & c_{3z}c_{3z} & c_{3z}c_{4z} \\ c_{4z}c_{1z} & c_{4z}c_{2z} & c_{4z}c_{3z} & c_{4z}c_{4z} \end{bmatrix}\{\Theta\}$$

$$+Nu(\Theta-\Theta_h)\frac{S}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$
=\frac{1}{20}V\begin{bmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{bmatrix}\frac{\{\Theta\}^{\tau+\Delta\tau}-\{\Theta\}^{\tau}}{\Delta\tau}
$$

$$
+V_x\frac{1}{24}\begin{bmatrix}c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\\c_{1x}&c_{2x}&c_{3x}&c_{4x}\end{bmatrix}\{\Theta\}
$$

$$
+V_y\frac{1}{24}\begin{bmatrix}c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\\c_{1y}&c_{2y}&c_{3y}&c_{4y}\end{bmatrix}\{\Theta\}
$$

$$
+V_z\frac{1}{24}\begin{bmatrix}c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\\c_{1z}&c_{2z}&c_{3z}&c_{4z}\end{bmatrix}\{\Theta\}
$$

$$
+\frac{1}{Pe}\frac{1}{36V}\begin{bmatrix}c_{1x}c_{1x}&c_{1x}c_{2x}&c_{1x}c_{3x}&c_{1x}c_{4x}\\c_{2x}c_{1x}&c_{2x}c_{2x}&c_{2x}c_{3x}&c_{2x}c_{4x}\\c_{3x}c_{1x}&c_{3x}c_{2x}&c_{3x}c_{3x}&c_{3x}c_{4x}\\c_{4x}c_{1x}&c_{4x}c_{2x}&c_{4x}c_{3x}&c_{4x}c_{4x}\end{bmatrix}\{\Theta\}
$$

$$
+\frac{1}{Pe}\frac{1}{36V}\begin{bmatrix}c_{1y}c_{1y}&c_{1y}c_{2y}&c_{1y}c_{3y}&c_{1y}c_{4y}\\c_{2y}c_{1y}&c_{2y}c_{2y}&c_{2y}c_{3y}&c_{2y}c_{4y}\\c_{3y}c_{1y}&c_{3y}c_{2y}&c_{3y}c_{3y}&c_{3y}c_{4y}\\c_{4y}c_{1y}&c_{4y}c_{2y}&c_{4y}c_{3y}&c_{4y}c_{4y}\end{bmatrix}\{\Theta\}
$$

$$
+\frac{1}{Pe}\frac{1}{36V}\begin{bmatrix}c_{1z}c_{1z}&c_{1z}c_{2z}&c_{1z}c_{3z}&c_{1z}c_{4z}\\c_{2z}c_{1z}&c_{2z}c_{2z}&c_{2z}c_{3z}&c_{2z}c_{4z}\\c_{3z}c_{1z}&c_{3z}c_{2z}&c_{3z}c_{3z}&c_{3z}c_{4z}\\c_{4z}c_{1z}&c_{4z}c_{2z}&c_{4z}c_{3z}&c_{4z}c_{4z}\end{bmatrix}\{\Theta\}
$$

$$+ Nu(\Theta - \Theta_h)\frac{S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= [C]\frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$
$$+ V_x[C_x]\{\Theta\} + V_y[C_y]\{\Theta\} + V_z[C_z]\{\Theta\}$$
$$+ \frac{1}{Pe}[S_{xx}]\{\Theta\} + \frac{1}{Pe}[S_{yy}]\{\Theta\} + \frac{1}{Pe}[S_{zz}]\{\Theta\}$$
$$+ Nu(\Theta - \Theta_h)\frac{S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= [C]\frac{\{\Theta\}^{\tau+\Delta\tau} - \{\Theta\}^{\tau}}{\Delta\tau}$$
$$+ (V_x[C_x] + V_y[C_y] + V_z[C_z])\{\Theta\}$$
$$+ \frac{1}{Pe}([S_{xx}] + [S_{yy}] + [S_{zz}])\{\Theta\}$$
$$+ Nu(\Theta - \Theta_h)\frac{S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$
\begin{aligned}
&[C]\frac{\{\Theta\}^{\tau+\Delta\tau}-\{\Theta\}^{\tau}}{\Delta\tau} \\
&+(V_x[C_x]+V_y[C_y]+V_z[C_z])\{\Theta\} \\
&+\frac{1}{Pe}([S_{xx}]+[S_{yy}]+[S_{zz}])\{\Theta\} \\
&+Nu(\Theta-\Theta_h)\frac{S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}
\end{aligned}
$$

陰解法

$$
\begin{aligned}
&\frac{[C]}{\Delta\tau}\{\Theta\}^{\tau+\Delta\tau} \\
&+(V_x[C_x]+V_y[C_y]+V_z[C_z])\{\Theta\}^{\tau+\Delta\tau} \\
&+\frac{1}{Pe}([S_{xx}]+[S_{yy}]+[S_{zz}])\{\Theta\}^{\tau+\Delta\tau} \\
&=\frac{[C]}{\Delta\tau}\{\Theta\}^{\tau}-Nu(\Theta-\Theta_h)\frac{S}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}
\end{aligned}
$$

境界条件

熱収支式には、主に第 1 種、第 3 種の境界条件が使用されます。